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INJECTIVE ENVELOPES OF BANACH MODULES

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1. Introduction. As in homology theory, the notion of injectivity was introduced in the category whose objects are Banach spaces and whose morphisms are contractive (i.e., of norm $\leqq 1$) linear maps, and the existence and uniqueness of the injective envelope of a Banach space was proved by H. B. Cohen [1] (cf. also [6]).

In the present paper we show that the corresponding statements are valid in the category whose objects are Banach modules over a Banach algebra and whose morphisms are contractive module homomorphisms, and that a flow (i.e., a compact Hausdorff space with a discrete group acting on it as onto homeomorphisms) has a projective cover. The latter, which seems to be, in a certain sense, a natural generalization of a result of A. M. Gleason [3; Theorem 3.2] (cf. 1° and Lemma 5 (i), (ii) below), is used to give a characterization of injective Banach modules over a discrete group algebra (Theorem 2 below). In the last section we are concerned with self-injective C^*-algebras (i.e., C^*-algebras which, considered as Banach modules over themselves, are injective).

Let A be a fixed Banach algebra with unit 1. We shall always assume that $||1|| = 1$. A unital left A-module X is called a left Banach A-module if its underlying vector space is a Banach space with the norm satisfying the condition:

$$||a \cdot x|| \leqq ||a|| \, ||x|| \quad \text{for} \quad a \in A \quad \text{and} \quad x \in X .$$

Similarly a right or two-sided Banach A-module is defined. But throughout this paper we shall exclusively treat left Banach A-modules unless otherwise specified, and abbreviate them to Banach A-modules. The letter X will denote a fixed but arbitrary Banach A-module.

DEFINITIONS. An extension of X is a pair (Y, κ) of a Banach A-module Y and an isometric module homomorphism $\kappa: X \rightarrow Y$. A Banach A-module X is injective if for each Banach A-module Y and each extension (Z, κ) of Y, any continuous module homomorphism $\alpha: Y \rightarrow X$ extends to a continuous module homomorphism $\hat{\alpha}: Z \rightarrow X$, i.e., $\hat{\alpha} \circ \kappa = \alpha$, with $||\hat{\alpha}|| = ||\alpha||$. An extension (Y, κ) of X is injective if Y is an injective