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POLYNOMIAL INVARIANTS OF ISOMETRY GROUPS OF INDEFINITE QUADRATIC LATTICES

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1. Introduction. There is a well known theorem of Chevalley [1] describing the ring of polynomial invariants of the Weyl group W of a complex simple Lie algebra $\mathfrak{g}$. Briefly the situation is this: Let the rank of $\mathfrak{g}$ be l and let Δ be the root system of $\mathfrak{g}$ with respect to some Cartan subalgebra. Then Δ spans an l dimensional real vector space $\boldsymbol{R}\Delta$ on which W acts as a finite linear group. By extension of the transpose action, W acts on the symmetric algebra $\mathscr{S}$ of the dual space $\boldsymbol{R}\Delta^*$. Chevalley's theorem says that the ring of W-invariant elements of $\mathscr{S}$ is generated by l algebraically independent homogeneous polynomials. The unique one of degree 2 is the quadratic form ψ on $\boldsymbol{R}\Delta$ which is due to the Killing form on $\mathfrak{g}$.

Initially we began to consider how the situation would change when Δ was an infinite root system defined by a non-singular symmetrizable Cartan matrix of non-finite type. The set-up is much the same, with ψ an indefinite quadratic form and W an infinite group acting in $\boldsymbol{R}\Delta$. The conclusions, however, are quite different, being of the form that ψ by itself generates the entire ring of invariants. Moreover it became clear that this type of result held in a considerably more general situation. If we recall that the integral span $\boldsymbol{Z}\Delta$ of Δ is a lattice in $\boldsymbol{R}\Delta$, then W is a subgroup of the group $O(\boldsymbol{Z}\Delta)$ of all isometries of $\boldsymbol{Z}\Delta$ with respect to ψ.

Suppose now that L is a lattice in a rational vector space V equipped with a non-degenerate indefinite quadratic form ψ. The type of result we obtain is that for all suitable subgroups G of the isometry group $O(L)$ of L, the ring of G-invariant polynomials on V is precisely $\boldsymbol{Q}[\psi]$. For example, this is true if $\dim V \geqq 3$ and G is any subgroup of finite index in $O(L)$ (Theorem 4.1). It is also true for a wide class of Weyl groups of infinite root systems, including all the hyperbolic root systems (Theorems 5.1 and 5.2) and we conjecture that it is in fact true for all Weyl groups arising from non-singular Cartan matrices of non-finite type.

At the center of the argument lies the celebrated theorem of Thue

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