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# FUBINI PRODUCTS OF $C^*$-ALGEBRAS

TADASI HURUYA

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**1. Introduction.** Let $C$ and $D$ be $C^*$-algebras and let $C \otimes D$ denote their minimal (or spatial) $C^*$-tensor product. For each $g \in C^*$ there is a unique bounded linear map $R_g$ of $C \otimes D$ to $D$ satisfying $R_g(c \otimes d) = \langle g, c\rangle d$. Similarly, for each $h \in D^*$ there is a unique bounded linear map $L_h$ of $C \otimes D$ to $C$ satisfying $L_h(c \otimes d) = \langle h, d\rangle c$. Let $A$ and $B$ be $C^*$-subalgebras of $C$ and $D$, respectively. We define the Fubini product of $A$ and $B$ with respect to $C \otimes D$ to be

$$F(A, B, C \otimes D) = \{x \in C \otimes D: R_g(x) \in B,\ L_h(x) \in A \text{ for every } g \in C^*,\ h \in D^*\}$$

(see [10]). If $C_1$, $C_2$ and $A$ are $C^*$-algebras such that $C_1 \supseteqq C_2 \supseteqq A$, and if $D_1$, $D_2$ and $B$ are $C^*$-algebras such that $D_1 \supseteqq D_2 \supseteqq B$, then $F(A, B, C_1 \otimes D_1)$ contains $F(A, B, C_2 \otimes D_2)$. In this paper we show that there is the largest Fubini product of $A$ and $B$, denoted by $A \otimes_F B$. We also consider a condition for a $C^*$-algebra to have property S [13]. Aided by [15], we give several Fubini products $A \otimes_F B$ strictly containing $A \otimes B$.

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**2. Some properties of Fubini products.** In this section we study certain elementary properties of Fubini products. The following result is known [12, Proposition 4.1] and is easy to check.

LEMMA 1. *Let $C$ and $D$ be $C^*$-algebras with $C^*$-subalgebras $A$ and $B$, respectively. Let* $\bar{C}$ and $\bar{D}$ *be the enveloping $W^*$-algebras of $C$ and $D$. Under the canonical embedding of $C \otimes D$ into the $W^*$-tensor product $\bar{C} \mathbin{\bar{\otimes}} \bar{D}$, let $\bar{A} \mathbin{\bar{\otimes}} \bar{B}$ denote the weak closure of $A \otimes B$. Then $F(A, B, C \otimes D)$ is just $(C \otimes D) \cap (\bar{A} \mathbin{\bar{\otimes}} \bar{B})$ and is a $C^*$-subalgebra of $C \otimes D$.*

LEMMA 2. *Let $A$, $C_1$ and $C_2$ be $C^*$-algebras such that $C_1 \supseteqq A$ and $C_2 \supseteqq A$, and let $B$, $D_1$ and $D_2$ be $C^*$-algebras such that $D_1 \supseteqq B$ and $D_2 \supseteqq B$. Suppose that there are four contractive and completely positive maps*:

$$\phi_1: C_1 \to C_2,\quad \phi_2: C_2 \to C_1,\quad \phi_i(a) = a \qquad (i = 1, 2,\quad a \in A)\ ,$$
$$\psi_1: D_1 \to D_2,\ \psi_2: D_2 \to D_1,\ \psi_i(b) = b \qquad (i = 1, 2,\quad b \in B)\ .$$