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INTERVAL MAPS, FACTORS OF MAPS, AND CHAOS[1]

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

JOSEPH AUSLANDER AND JAMES A. YORKE

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Abstract. We investigate the dynamical properties of continuous maps of a compact metric space into itself. The notion of chaos is defined as the instability of all trajectories in a set together with the existence of a dense orbit. In particular we show that any map on an interval satisfying a generalized period three condition must have a nontrivial (uncountable) minimal set as well as "large" chaotic subsets. The nontrivial minimal sets are investigated by lifting to sequence spaces while the chaotic sets are investigated using "factors," projections of larger spaces onto smaller ones.

Introduction. There has been an increasing realization that the dynamics behind many physical processes are inherently chaotic. Just to cite an example, consider the fact that Landau [1] explained the transition to turbulence for flow past a solid object in terms that invoked an infinite number of degrees of freedom. Then Ruelle and Takens [2] showed that similar processes could lead to even more chaotic behavior while requiring only five degrees of freedom, i.e., five dimensions. Now it is recognized that only three degrees of freedom are required. Lorenz [3], Rossler [4, 5], Curry and Yorke [6] and Bowen [7] are just some of the papers which discuss the physical relevance of three dimensional chaotic flows or two dimensional chaotic diffeomorphisms. May [8] and Li and Yorke [9] emphasize the physical significance of continuous chaotic maps on the real line. A number of concepts have been introduced to describe the relevant phenomena. Most important is ergodic theory but this is particularly difficult to apply to particular situations; the simplest nonlinear dynamical process is the iteration of a quadratic map on an interval, $T(x) = rx(1 - x)$, and yet even here the situation is unsatisfactory: The set of r for which there is an absolutely continuous invariant probability measure has only recently been shown to be infinite (Pianigiani [10] and Jacobson and Sinai) yet it seems likely that this set

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