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ON EXTREMAL QUASICONFORMAL MAPPINGS COMPATIBLE WITH A FUCHSIAN GROUP

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1. Introduction. Let U be the upper half-plane and let $\hat{\boldsymbol{R}} = \boldsymbol{R} \cup \{\infty\}$ be the extended real line. We denote by $PSL(2, \boldsymbol{R})$ the real Möbius group, that is, the group of all the conformal automorphisms of U. A discrete subgroup G of $PSL(2, \boldsymbol{R})$ is called a Fuchsian group. The limit set $\Lambda(G)$ of a Fuchsian group G is the derived set of the set which consists of all the images $\gamma(i)$ of the point $z = i$ under $\gamma \in G$. We say that a Fuchsian group G is non-elementary whenever $\Lambda(G)$ contains more than two points. A Fuchsian group G is said to be of the first kind if $\Lambda(G)=\hat{\boldsymbol{R}}$; G is said to be of the second kind if $\Lambda(G)\neq\hat{\boldsymbol{R}}$. It is well-known that, if G is a non-elementary Fuchsian group of the second kind, then $\Lambda(G)$ is a nowhere dense perfect subset of $\hat{\boldsymbol{R}}$, which is invariant under G.

Let G be a Fuchsian group and let σ be a closed subset of $\hat{\boldsymbol{R}}$, which is invariant under G and which contains at least three points. We define $\Sigma(G)$ as the family which consists of all such σ. As is known, every σ in $\Sigma(G)$ contains $\Lambda(G)$. Let f be a quasiconformal automorphism of U, which is compatible with G: that is, $fGf^{-1} \subset PSL(2, \boldsymbol{R})$. All such f form a family $F(G)$. It is known that every f in $F(G)$ is extensible to a homeomorphism of $U \cup \hat{\boldsymbol{R}}$, which is also denoted by the same letter f. For $f \in F(G)$ and $\sigma \in \Sigma(G)$, we define $F(G, f, \sigma)$ as the set of all the $g \in F(G)$ satisfying $g|_\sigma = f|_\sigma$, where $g|_\sigma$ means the restriction of g to σ. We put

$$k(G, f, \sigma) = \inf \|\mu_g\| , \tag{1.1}$$

where $\|\mu_g\|$ means the L_∞ norm of the Beltrami coefficient $\mu_g = g_{\bar{z}}/g_z$ of g and the infimum is taken over all $g \in F(G, f, \sigma)$. By means of a normal family argument of quasiconformal mappings, we can check that there exists some $g \in F(G, f, \sigma)$ with $\|\mu_g\| = k(G, f, \sigma)$ (see [6]). Such a mapping g is said to be extremal in the class $F(G, f, \sigma)$.

Let Γ be a subgroup of a Fuchsian group G. By definition, it is obvious that $\Sigma(G) \subset \Sigma(\Gamma)$, $F(G) \subset F(\Gamma)$ and that $F(G, f, \sigma) \subset F(\Gamma, f, \sigma)$ for every $f \in F(G)$ and every $\sigma \in \Sigma(G)$. Thus, by (1.1), clearly we have

$$k(G, f, \sigma) \geqq k(\Gamma, f, \sigma) \tag{1.2}$$