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THE PRODUCT OF OPERATORS WITH CLOSED RANGE AND AN EXTENSION OF THE REVERSE ORDER LAW

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1. Introduction. Let A be a (bounded linear) operator on a Hilbert space H. If A has closed range, then there is a unique operator $A^\dagger$ called the Moore-Penrose inverse or generalized inverse of A, which satisfies the following four identities [2, p. 321]:

$$AA^\dagger A = A\,, \qquad A^\dagger AA^\dagger = A^\dagger\,, \qquad (A^\dagger A)^* = A^\dagger A \qquad \text{and}$$
$$(AA^\dagger)^* = AA^\dagger\,.$$

We denote by (CR) the set of all operators on H with closed range (or equivalently, that of all operators with Moore-Penrose inverses). For two operators A and B in (CR), one problem is to find the condition under which the product AB is in (CR). Bouldin [3] [5] gave a geometric characterization of the condition in terms of the angle between two linear subspaces, and recently Nikaido [16] showed a topological characterization of it (for Banach space operators). Another problem is to represent the Moore-Penrose inverse $(AB)^\dagger$ in a reasonable form, that is, to generalize the reverse order law $(AB)^{-1} = B^{-1}A^{-1}$ for invertible operators. Many authors [1], [4], [6], [9], [10], [18]-[20], etc. (some of them in the setting of matrices) studied this problem. Barwick and Gilbert [1], Bouldin [4] [6], Galperin and Waksman [9], etc. proved some necessary and sufficient conditions which guarantee the "generalized" reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ holds.

In this paper we shall treat the product of two operators with closed range. In Section 2 we shall show some norm inequalities for the product to have closed range, which enable us to refine the results in [3] and [16]. In Section 3, using our result in [12], we shall present an extension of the (generalized) reverse order law, and extend some main results in [1], [4], [6] and [9].

Throughout this paper all operators are bounded linear. A projection is a selfadjoint idempotent operator, and it is an orthogonal projection onto a closed linear subspace of H. For projections P and Q onto the closed linear subspaces M and N, we write, in lattice theoretic notations, $P^\perp$, $P \vee Q$ and $P \wedge Q$ the projections onto the orthocomplement $M^\perp$ of