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THE FIRST EIGENVALUE OF THE LAPLACIAN ON TWO DIMENSIONAL RIEMANNIAN MANIFOLDS

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1. Introduction. Let M be a two dimensional compact Riemannian manifold without boundary. Let w be a fixed point on M. For any sufficiently small $\varepsilon > 0$, let B_ε be the geodesic disk of radius ε with the center w. We put $M_\varepsilon = M \setminus \bar{B}_\varepsilon$. Let $\lambda_1(\varepsilon)$ be the first positive eigenvalue of the Laplacian $\Delta = -\mathrm{div}\,\mathrm{grad}$ in M_ε under the Dirichlet condition on ∂B_ε.

The main result of this paper is the following:

THEOREM 1. *Assume $n = 2$. Then*

$$\lambda_1(\varepsilon) = -2\pi |M|^{-1} (\log \varepsilon)^{-1} + O((\log \varepsilon)^{-2}) \tag{1.1}$$

holds as ε tends to zero. Here $|M|$ denotes the area of M.

Chavel-Feldman [3] showed that $\lambda_1(\varepsilon) \to 0$ as ε tends to zero. Theorem 1 improves their result for the case $n = 2$. The readers may also refer to Matsuzawa-Tanno [5] where the case $M = (S^2$, the standard metric) was studied.

In §2, we give the Schiffer-Spencer variational formula for the resolvent kernels of the Laplacian with the Dirichlet condition on the boundary. For the Schiffer-Spencer formula, the reader may refer to Schiffer-Spencer [6] and Ozawa [7]. In [7], the author gave an asymptotic formula for the j-th eigenvalue of the Laplacian when we cut off a small ball of radius ε from a given bounded domain in $\boldsymbol{R}^n$ $(n = 2, 3)$. In §3, we prove Theorem 1. In §4, we make a remark on the inequality of Cheeger.

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2. A variant of the Schiffer-Spencer formula. Let $L^2(M)$ (resp. $L^2(M_\varepsilon)$) denote the Hilbert space of square integrable functions on M (resp. M_ε). By A we denote the self-adjoint operator in $L^2(M)$ associated with the Laplacian on M. Let $A(\varepsilon)$ denote the self-adjoint operator in $L^2(M_\varepsilon)$ associated with the Laplacian in M_ε under the Dirichlet condition