Tôhoku Math. Journ.
34 (1982), 495-498.

A CRITERION FOR UNIFORM INTEGRABILITY OF EXPONENTIAL MARTINGALES

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

TATSUYA OKADA

(Received November 28, 1981, revised April 17, 1982)

Let (Ω, F, P) be a complete probability space equipped with a non-decreasing right continuous family (F_t) of sub σ-fields of F such that F_0 contains all null sets. We shall use the notations given in Meyer [5]. Let M be a local martingale with $M_0 = 0$, M^c its continuous part and $\langle M^c \rangle$ the increasing process associated with M^c. We put $\Delta M_{\cdot} = M_{\cdot} - M_{\cdot -}$ and assume the condition $\Delta M_{\cdot} > -1$ throughout this note. Denote the exponential martingale of M by $\mathscr{E}(M)$, that is, $\mathscr{E}(M)_t = \exp\{M_t - (1/2)\langle M^c \rangle_t + (\log(1+x) - x) \cdot \mu_t\}$, where μ is the integer valued random measure associated with jumps of M. As is well-known, $\mathscr{E}(M)$ is a positive supermartingale with $\mathscr{E}(M)_0 = 1$ but it is not always a uniformly integrable martingale. Girsanov [1] raised the problem of finding a sufficient condition for the process $\mathscr{E}(M)$ to be a uniformly integrable martingale. The purpose of this paper is to establish the following.

THEOREM. *If, for some α with $0 \leqq \alpha < 1$ and a non-negative constant C,*

$$(1) \qquad (\exp\{\alpha M_S + ((1/2) - \alpha)\langle M^c \rangle_S - (1-\alpha) C \langle M^c \rangle_S^{1/2} + (\log(1+x) - x + (1-\alpha) x^2/(1+x)) \cdot \mu_S\})_{S \in \mathscr{S}_b}$$

is uniformly integrable, then $\mathscr{E}(M)$ is a uniformly integrable martingale. Here $\mathscr{S}_b$ denotes the set of all bounded stopping times.

REMARK 1. The above theorem is an improvement of the results in Novikov [6], [8], Kazamaki [2], and Lépingle and Mémin [4]. For example, our theorem implies the result in [8] (resp. [4]) in the case of $\Delta M = 0$ and $\alpha = 1/2$ (resp. $C = 0$).

REMARK 2. Let $\tilde{M} = M - (\langle M^c \rangle - C \langle M^c \rangle^{1/2}) - (x^2/(1+x)) \cdot \mu$ and $A^{(\alpha)} = \log \mathscr{E}(M) - (1-\alpha)\tilde{M}$. If $\{\exp(A_S^{(\alpha)})\}_{S \in \mathscr{S}_b}$ is uniformly integrable for some α with $0 \leqq \alpha < 1$, then so is $\{\exp(A_S^{(\beta)})\}_{S \in \mathscr{S}_b}$ for every β with $\alpha < \beta < 1$. Indeed, letting $S \in \mathscr{S}_b$, we have