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NONSELFADJOINT SUBALGEBRAS ASSOCIATED WITH COMPACT ABELIAN GROUP ACTIONS ON FINITE VON NEUMANN ALGEBRAS

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1. Introduction. Let G be a compact abelian group whose dual Γ has a total order. Suppose that M is a von Neumann algebra with a faithful normal tracial state τ and $\{\alpha_g\}_{g\in G}$ is a σ-weakly continuous representation of G as *-automorphisms of M such that $\tau\circ\alpha_g = \tau$, $g\in G$. Put $\Gamma_+ = \{\gamma\in\Gamma: \gamma \geqq 0\}$ and let $H^\infty(\alpha)$ be the set of $x\in M$ such that $Sp_\alpha(x)\subset\Gamma_+$. Recently, the structure of $H^\infty(\alpha)$ has been investigated by several authors (cf. [7], [8], [9], [10], [12], [13], [15]). It is well-known that $H^\infty(\alpha)$ is a finite maximal subdiagonal algebra of M (cf. [8]). However, $H^\infty(\alpha)$ is not necessarily maximal as a σ-weakly closed subalgebra of M. McAsey, Muhly and the author in [9], [10] and [15] studied the maximality of typical examples of $H^\infty(\alpha)$ which are called nonselfadjoint crossed products.

Our aim in this paper is to investigate the maximality of $H^\infty(\alpha)$ as a σ-weakly closed subalgebra of M. Our method is based on a characterization of spectral subspaces and the invariant subspace structure of the noncommutative Lebesgue space $L^2(M, \tau)$ associated with M and τ in the sense of Segal [16]. In §2, we give a characterization of spectral subspaces. For every $\gamma\in\Gamma$, we put $M_\gamma = \{x\in M: \alpha_g(x) = \langle g, \gamma\rangle x, g\in G\}$. Suppose that the center $\mathfrak{Z}(M_0)$ of M_0 is contained in the center $\mathfrak{Z}(M)$ of M. If $M_\gamma \neq \{0\}$, then there is a partial isometry u_γ in M_γ and a projection e_γ in $\mathfrak{Z}(M_0)$ such that $M_\gamma = M_0u_\gamma$ and $u_\gamma^*u_\gamma = u_\gamma u_\gamma^* = e_\gamma$. In particular, if M_0 is a factor, then we may choose a unitary element u_γ in M_γ such that $M_\gamma = M_0u_\gamma$. In §3, we first define the cocycles of canonical left-invariant subspaces of $L^2(M, \tau)$. If M_0 is a factor, then every two-sided invariant subspace is left-pure and left-full. As the main result in this paper, we show that, if $\mathfrak{Z}(M_0)\subset\mathfrak{Z}(M)$ and if there is no nonzero projection p of $\mathfrak{Z}(M_0)$ with $Mp = M_0p$, then $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if M_0 is a factor and $Sp\alpha$ is a subgroup (of Γ) with an archimedean order.

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