Tôhoku Math. Journ.
36 (1984), 49–74.

ON JOSEPH'S CONSTRUCTION OF WEYL GROUP REPRESENTATIONS

RYOSHI HOTTA

(Received January 31, 1983)

One of the purposes of this paper is to identify a new construction of representations of Weyl groups due to Joseph [6] with Springer's construction (see [10], [11]). More precisely, let O be a nilpotent orbit of a complex semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ the Lie algebra of a maximal unipotent subgroup. To each irreducible component U of $O \cap \mathfrak{n}$, Joseph has attached a certain polynomial p_U on the dual of a Cartan subalgebra and has shown that the p_U's span a W-submodule (W denotes the Weyl group) in the space of polynomials when U runs through the irreducible components of $O \cap \mathfrak{n}$. On the other hand, for a nilpotent $A \in O$, let $\mathscr{B}^A = \{gB \mid g^{-1}A \in \mathfrak{n}\}$ be the S3 variety, where $B = N_G(\mathfrak{n})$ is the Borel subgroup whose unipotent radical has the Lie algebra $\mathfrak{n}$. Springer [10] defined W-module structures on the rational homology groups $H_*(\mathscr{B}^A, \boldsymbol{Q})$ on which also the finite group $C(A) = Z_G(A)/Z_G(A)^\circ$ acts compatibly. The $C(A)$-fixed subspace $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})^{C(A)}$ of the top homology ($d(A) = \dim \mathscr{B}^A$) is known to be W-irreducible. In this note, it will be proved (Theorem 3) that this irreducible W-module is isomorphic (up to the sign representation) to the previous W-module $\sum_U \boldsymbol{Q} p_U$ defined by Joseph. As Joseph has pointed out, it follows from the above identification that the polynomials p_U are harmonic. Furthermore, Spaltenstein [9] has shown that there is a natural surjection σ from the set $I(\mathscr{B}^A)$ of the irreducible components of $\mathscr{B}^A$ onto the set $I(O \cap \mathfrak{n})$ of those of $O \cap \mathfrak{n}$. The above isomorphism is given by the correspondence

$$p_U \mapsto \sum_{C \in \sigma^{-1}(U)} [C] \in H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})^{C(A)}$$

where $[C]$ is the fundamental cycle for an irreducible component C of $\mathscr{B}^A$.

In order to prove the above identification, in §2, we shall rather extend Joseph's idea to obtain a W-module isomorphic to the full homology group $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$. For this, we take the universal covering space $\tilde{O}$ of O and consider $\rho^{-1}(O \cap \mathfrak{n}) \subset \tilde{O}$ ($\rho: \tilde{O} \to O$). We define a W-module

Supported in part by the Grants-in-Aid for Scientific and Co-operative Research, the Ministry of Education, Science and Culture, Japan.