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A MODIFIED FORM OF THE VARIATION-OF-CONSTANTS FORMULA FOR EQUATIONS WITH INFINITE DELAY

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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1. Introduction. For equations with finite delay, the variation-of-constants formula was given in Halanay's book [2]. Banks [1] pointed out a mistake in this book and presented the correct result. Since equations with finite delay were mainly considered, the results were derived under the restrictive hypotheses: the kernel function $\eta(t, \theta)$ of the linear operator $L(t, \cdot)$ (cf. Theorem 2.1) is constant for sufficiently small $\theta < 0$.

In the present paper, we start from the following hypotheses: $L(t, \phi)$ is continuous and the phase space for ϕ is the general space for equations with infinite delay introduced by Hale and Kato [4]. From the first hypothesis, the Borel measurability of $\eta(t, \theta)$ is naturally induced; from the second, the constant property of $\eta(t, \theta)$ mentioned above cannot be assumed (see Theorem 2.1). The equation related to the fundamental matrix is reduced to the standard equation with infinite delay (Proposition 3.1):

$$x'(t) = L(t, x_t) + h(t) , \tag{1.1}$$

where h is locally integrable. The representation of solutions in Theorem 3.3, which is already announced in [5], has a form that is somewhat different from the variation-of-constants formula given in [1], [2], [3]. For the special phase space $\mathscr{C}_\gamma$ defined in Section 4, our formula is rewritten in a form analogous to the variation-of-constants formula. However, it contains a new term depending on the "exponential limit of the initial function at $-\infty$". Finally, we remark that the present result is an extension of the work for autonomous equations [6] to the case of nonautonomous equations.

2. Representation of linear operators. For a function $x: (-\infty, a) \to C^n$, let $x_t: (-\infty, 0] \to C^n$, $t < a$, be defined by $x_t(\theta) = x(t + \theta)$ for θ in $(-\infty, 0]$. Suppose $\mathscr{B}$ is a linear space of functions $\phi, \psi, \cdots$, mapping $(-\infty, 0]$ into C^n, with a semi-norm $|\phi|, |\psi|, \cdots$ having the following