SOME REMARKS ON THE INSTABILITY FLAG

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Introduction. Let $G$ be a reductive group acting on a projective variety $M \subset P(V)$. Mumford raised the question of associating a canonical parabolic subgroup $P(m)$ of $G$ to any nonsemistable point $m \in M$ ([9, p. 64]). This would have the interesting consequence that if the action of $G$ as well as the point $m$ are rational over a perfect field, then $P(m)$ would (by Galois descent) be rational. Recently this problem was solved in the affirmative by Kempf [7] and Rousseau [16].

Besides, one can also associate to $m$, a conjugacy class of 1-parameter subgroups (1-PS's) in $P(m)$. Let $\lambda$ be a 1-PS in this class and let $V = \bigoplus_{i \in Z} V_i$ be the decomposition of $V$ where $V_i$ is the space of weight vectors of weight $i$ for $\lambda$. Let $m = m_0 + m_1$ with $0 \neq m_0 \in V_j$ and $m_1 \in \bigoplus_{i > j} V_i$. Then we show by a refinement of Kempf’s arguments that $P(m_0) = P(m)$ (Proposition 1.9). Moreover for the natural action of $P/U$ on $V_j$, where $U$ is the unipotent radical of $P$, $m_0$ becomes semistable after the polarisation is replaced by a multiple and the action of $P/U$ is twisted by a dominant character (Proposition 1.12).

In Section 2 we investigate the existence of the instability flag over non-perfect fields. We prove that if $G$ acts separably (see Definition 2.1) on $M$ over $k$ and $m$ is a $k$-rational nonsemistable point then $P(m)$ is defined over $k$ (Theorem 2.3).

In Section 3 we apply these results to the study of bundles on projective nonsingular varieties. If $E$ is a semistable vector bundle on a projective curve $X$ defined over an algebraically closed field of characteristic 0 then it follows from the characterisation of stable bundles in terms of unitary representation of Fuchsian groups ([10]) that any bundle associated to $E$ by extension of structure group is also semistable. An analogous result is also valid for $G$-bundles, thanks to [14]. An algebraic proof of this has been given in [5], [8]. We give another algebraic proof using the existence of parabolic subgroups mentioned above. Our ideas are close to those in [1].

We will now give an outline of our proof. For simplicity we assume that $X$ is a curve. A principal $G$-bundle $E \to X$ is semistable if for any reduction of structure group to a parabolic subgroup $P$ and any dominant