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SOME REMARKS ON THE INSTABILITY FLAG

S. RAMANAN AND A. RAMANATHAN

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Introduction. Let G be a reductive group acting on a projective variety $M \subset \boldsymbol{P}(V)$. Mumford raised the question of associating a canonical parabolic subgroup $P(m)$ of G to any nonsemistable point $m \in M$ ([9, p. 64]). This would have the interesting consequence that if the action of G as well as the point m are rational over a perfect field, then $P(m)$ would (by Galois descent) be rational. Recently this problem was solved in the affirmative by Kempf [7] and Rousseau [16].

Besides, one can also associate to m, a conjugacy class of 1-parameter subgroups (1-PS's) in $P(m)$. Let λ be a 1-PS in this class and let $V = \bigoplus_{i \in \boldsymbol{Z}} V_i$ be the decomposition of V where V_i is the space of weight vectors of weight i for λ. Let $m = m_0 + m_1$ with $0 \neq m_0 \in V_j$ and $m_1 \in \bigoplus_{i > j} V_i$. Then we show by a refinement of Kempf's arguments that $P(m_0) = P(m)$ (Proposition 1.9). Moreover for the natural action of P/U on V_j, where U is the unipotent radical of P, m_0 becomes semistable after the polarisation is replaced by a multiple and the action of P/U is twisted by a dominant character (Proposition 1.12).

In Section 2 we investigate the existence of the instability flag over non-perfect fields. We prove that if G acts separably (see Definition 2.1) on M over k and m is a k-rational nonsemistable point then $P(m)$ is defined over k (Theorem 2.3).

In Section 3 we apply these results to the study of bundles on projective nonsingular varieties. If E is a semistable vector bundle on a projective curve X defined over an algebraically closed field of characteristic 0 then it follows from the characterisation of stable bundles in terms of unitary representation of Fuchsian groups ([10]) that any bundle associated to E by exension of structure group is also semistable. An analogous result is also valid for G-bundles, thanks to [14]. An algebraic proof of this has been given in [5], [8]. We give another algebraic proof using the existence of parabolic subgroups mentioned above. Our ideas are close to those in [1].

We will now give an outline of our proof. For simplicity we assume that X is a curve. A principal G-bundle $E \to X$ is semistable if for any reduction of structure group to a parabolic subgroup P and any dominant