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# LOWER BOUNDS FOR THE EIGENVALUES OF THE FIXED VIBRATING MEMBRANE PROBLEMS

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**1. Introduction.** Let $\Omega$ be a bounded domain of the Euclidean space $\boldsymbol{R}^n$ with appropriately regular boundary $\partial\Omega$. We consider the classical fixed vibrating membrane problem:

$$\Delta u = \lambda u \text{ on } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega .$$

Here $\Delta$ is the standard Laplacian $-\sum_{i=1}^{n} \partial^2/\partial(x_i)^2$ of the Euclidean space $\boldsymbol{R}^n$. Let $\{\lambda_1 < \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \cdots \uparrow \infty\}$ be the eigenvalues of this problem counted with their multiplicities.

G. Pólya conjectured (cf. [8])

$$\lambda_k \geqq C_n \operatorname{Vol}(\Omega)^{-2/n} k^{2/n} \quad \text{for every} \quad k , \tag{1.1}$$

which was proved by him in case of *space-covering domains* $\Omega$. That is, an infinity of domains congruent to $\Omega$ cover the whole space $\boldsymbol{R}^n$ without gaps and without overlapping except a set of measure zero. Here the positive constant $C_n$ is $4\pi^2\omega_n^{-2/n}$, $\omega_n = \pi^{n/2}/\Gamma((n/2)+1)$ is the volume of the unit ball and $\operatorname{Vol}(\Omega)$ is the volume of $\Omega$. The conjecture of Pólya is closely related to H. Weyl's asymptotic formula (cf. [10])

$$\lambda_k \sim C_n \operatorname{Vol}(\Omega)^{-2/n} k^{2/n} \quad \text{as} \quad k \to \infty , \tag{1.2}$$

which shows the sharpness of Pólya's bounds for higher eigenvalues.

E. H. Lieb [5] has showed that (1.1) is true when $C_n$ is replaced by a smaller constant $D_n^{-2/n}$ where $D_3^{-2/3} = C_3 \times 0.2773$ and $D_3 = 0.1156$. Recently S. Y. Cheng and P. Li (cf. [11, p. 22]) showed

$$\lambda_k \geqq A_n \operatorname{Vol}(\Omega)^{-2/n} k^{2/n} \quad \text{for every} \quad k , \tag{1.3}$$

which is valid for general compact riemannian manifold with smooth boundary. Here the constant $A_n$ is $2cn^{-1}e^{-2/n}$, $c = c'^2\ ((n-2)/(2n-2))^2$ and $c'$ is the Sobolev constant $n\omega_n^{1/n}$ which satisfies the inequality $\operatorname{Vol}(\partial\Omega)^n \geqq c'^n \operatorname{Vol}(\Omega)^{n-1}$. It should be noted that the constant $A_n$ is asymptotically $e^2 2^{-1} n^{-1}$ as $n \to \infty$.

In this paper, we show the following:

THEOREM 1. *For every eigenvalue* $\lambda_k$ *of the fixed vibrating membrane*