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REFLECTION GROUPS AND THE EIGENVALUE PROBLEMS OF VIBRATING MEMBRANES WITH MIXED BOUNDARY CONDITIONS

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Introduction. Throughout this paper, (M, g) is an n-dimensional space form of constant curvature, that is, the Euclidean space $\boldsymbol{R}^n$, the standard sphere $\boldsymbol{S}^n$ or the hyperbolic space $\boldsymbol{H}^n$. Let Δ be the (non-negative) Laplacian of (M, g). Let Ω be a bounded domain in M with an appropriately regular boundary $\partial\Omega$. For an arbitrary fixed real number ρ, let us consider the following boundary value eigenvalue problem:

$$\begin{cases} \Delta f = \lambda f & \text{in } \Omega, \\ f = 0 & \text{on } \Gamma_1, \quad \text{and} \\ \partial f/\partial \boldsymbol{n} = \rho f & \text{a.e. } \Gamma_2, \text{ i.e., where the exterior normal } \boldsymbol{n} \text{ of } \Gamma_2 \text{ is defined.} \end{cases}$$

Here the boundary $\partial\Omega$ is a disjoint union of Γ_1 and Γ_2. It is called (cf. [B, p. 91]) to be

(D) the *fixed membrane* problem if $\Gamma_2 = \emptyset$,

(N) the *free membrane* problem if $\Gamma_1 = \emptyset$, or

(M_ρ) the membrane problem of *mixed boundary conditions* if $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$.

It is well known that each problem has a discrete spectrum of the eigenvalues with finite multiplicity. We denote by $\mathrm{Spec}_D(\Omega)$, $\mathrm{Spec}_N(\Omega)$ and $\mathrm{Spec}_{M_\rho}(\Omega)$, the spectra of the problems (D), (N) and (M_ρ), respectively.

One of the important problems of the spectra is to research how the spectra $\mathrm{Spec}_D(\Omega)$, $\mathrm{Spec}_N(\Omega)$ or $\mathrm{Spec}_{M_\rho}(\Omega)$ reflect the shape of Ω. In his paper [K], M. Kac posed the following problem:

For two bounded domains $\Omega, \tilde{\Omega}$ in $\boldsymbol{R}^n$ ($n \geqq 2$), assume that $\mathrm{Spec}_D(\Omega) = \mathrm{Spec}_D(\tilde{\Omega})$. *Are the domains $\Omega, \tilde{\Omega}$ congruent in $\boldsymbol{R}^n$?*

Here two domains $\Omega, \tilde{\Omega}$ are congruent in the space form (M, g) if there exists an isometry Φ of (M, g) such that $\Phi(\Omega) = \tilde{\Omega}$. Note that $\Omega, \tilde{\Omega}$ are isometric with respect to the induced metrics from (M, g) if and only if they are congruent in (M, g) because of simple connectedness of M (cf. [K.N., p. 252]).