Tôhoku Math. Journ.
36 (1984), 407-426.

TOWARDS AN ALGEBRO-GEOMETRIC INTERPRETATION OF THE NEUMANN SYSTEM

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(Received September 7, 1983)

Abstract. Lax equations and constants of motion for C. Neumann's system of constrained harmonic oscillators are derived in a systematic way from the Burchnall-Chaundy-Krichever theory of 2nd-order differential operators $D^2+q(t)$. The approach is based on a geometric step: to map the algebraic curve and linebundle associated with $D^2+q(t)$ to a larger projective space by means of a suitable linear system. The image of $D^2+q(t)$ is, roughly speaking, just the Lax operator for the Neumann system.

1. Introduction. In the 1920's, J. L. Burchnall, T. W. Chaundy, and H. F. Baker studied differential operators $L = D^n + q_2(t)D^{n-2} + \cdots + q_n(t)$ $(D = (d/dt))$ that commute with at least one other differential operator B of some order m relatively prime to n [2 − 5]. They realized that the commutant

$$\bar{L} = \{\text{diff. ops. } A \mid [L, A] = 0\}$$

of such an L has the structure of the affine coordinate ring of an algebraic curve $\mathscr{R}$,

$$\bar{L} \cong C[x_1, \cdots, x_r]/\text{ideal} ,$$

and, especially in Baker's paper [2], that the determination of the operator L from the curve $\mathscr{R}$ requires some additional data: most important among those, when $\mathscr{R}$ is a smooth affine curve, is a linebundle with a one-dimensional space of holomorphic sections. From a suitable section, one constructs an eigenfunction ψ of L, and then recovers L itself.

These remarkable results were largely forgotten, and eventually rediscovered, with improvements, by Krichever [13] in the late 1970's. His motivation was the recently established connection between commuting differential operators and soliton solutions of integrable partial differential equations, such as the Korteweg-de Vries equation [9, 11, 13, 14, 17]. In the last few years, the relation between linebundles over algebraic curves and special differential operators has led to the discovery of many amazing "coincidences". In this paper, I will explain one such "coincidence", the occurrence of integrable constrained oscillator systems (par-