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WEAK SOLUTIONS OF NAVIER-STOKES EQUATIONS

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Introduction. Consider the initial-value problem for the Navier-Stokes equation in a domain Ω of $\boldsymbol{R}^n$:

$$\text{(N-S)} \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u\cdot\nabla u + \nabla p = f\,; \quad \nabla\cdot u = 0\,, \quad x \in \Omega\,, \quad 0 < t < T\,. \\ u|_\Gamma = 0\,; \quad u|_{t=0} = a \end{cases}$$

(Γ: the boundary of Ω) where $u = u(x, t)$ is the unknown velocity vector $(u^1, u^2, \cdots, u^n)$; $p = p(x, t)$ is the unknown pressure; $a = a(x)$ is the initial velocity vector field; $f = f(x, t)$ is a given external force. Here we use the notation:

$$u\cdot\nabla v = \sum_{i=1}^n u^i \frac{\partial v}{\partial x_i}\,; \quad \nabla\cdot u = \sum_{i=1}^n \frac{\partial u^i}{\partial x_i}$$

for vector functions u, v.

In his famous paper [8], E. Hopf showed the existence of the so-called Hopf's weak solution to the problem (N-S). The first purpose of the present paper is to show the existence of a weak solution, belonging to some class of functions introduced by J. L. Lions [14], which seems to have a somewhat stronger property than the Hopf's weak solution.

In the general case the uniqueness of a weak solution has been not known. Lions-Prodi [15] gave the uniqueness theorem when $n = 2$. C. Foias [15] introduced function spaces $L^{r,r'}$ (for the definition see the chapter 1 of this paper), and showed that if $\Omega = \boldsymbol{R}^n$, and if there is a weak solution u in $L^{r,r'}$ with $r > n$, and with $n/r + 2/r' < 1$, then this u is the only weak solution of (N-S). J. Serrin [23] gave a similar theorem under the assumptions that Ω is a general domain of $\boldsymbol{R}^n$ ($n = 2, 3, 4$), and that a pair of exponents r, r' satisfies $r > n$ and $n/r + 2/r' \leqq 1$. The second purpose is to generalize the Foias-Serrin uniqueness theorem in two directions. First we shall remove the artificial restriction on the dimension n imposed in the theorem of Serrin. Secondly, we shall show that if there is a weak solution u in $L^{n,\infty}$ which is right continuous for t as an L^n-valued function, then u is the only weak solution. Recently von Wahl [26] obtained similar results (the uniqueness in the class $C([0, T); L^n)$) under the assumptions that the initial velocity and the external force