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A FORMULA IN SIMPLE JORDAN ALGEBRAS

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0. In this paper, we give a proof of a formula ((14) in 3) which gives a useful parametrization in reduced simple Jordan algebras. We also summarize some relevant facts on Jordan algebras. This formula and some of its consequences (e.g. Prop. 4, 5) were already used in [4b] and [5]. For basic facts on Jordan algebras, the reader is referred to [2], [3], [4a, c] and [6].

Let A be a Jordan algebra over a field F of characteristic zero. We use the following notation:

$$\{a, b, c\} = (ab)c + a(bc) - b(ac) ,$$
$$T_a(x) = ax , \qquad P_a(x) = \{a, x, a\} = (2T_a^2 - T_{a^2})x ,$$
$$(a \square b)x = \{a, b, x\} = (T_{ab} + [T_a, T_b])x \qquad (a, b, c, x \in A) .$$

It is well-known that A has a structure of "JTS" with respect to this triple product $\{\ \ \}$, i.e. one has

$$(1) \qquad \{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\} .$$

Throughout this paper, we assume that A is simple (and semi-simple). Then A has a unit element 1 and the following symmetric bilinear form on A is non-degenerate:

$$(2) \qquad \langle x, y\rangle = \kappa\, \mathrm{tr}(x \square y) = \kappa\, \mathrm{tr}(T_{xy}) \qquad (x, y \in A) ,$$

where κ is a fixed element in $F^\times(=F - \{0\})$.

1. Let e be an idempotent in A and let

$$A_\lambda = A_\lambda(e) = \{x \in A \mid ex = \lambda x\} \qquad \text{for} \quad \lambda \in F .$$

Then one has the direct sum decomposition ("Peirce decomposition")

$$A = A_0 + A_{1/2} + A_1$$

with

$$(3) \qquad \begin{cases} A_\lambda^2 = A_\lambda , \qquad A_\lambda A_{1/2} \subset A_{1/2} \qquad (\lambda = 0, 1) , \\ A_0A_1 = 0 , \qquad A_{1/2}^2 \subset A_0 + A_1 , \\ \{A_\lambda, A_\mu, A_\nu\} \subset A_{\lambda-\mu+\nu} . \end{cases}$$