Tôhoku Math. Journ.
36 (1983), 591–609.

RATIONAL MAPPINGS OF DEL PEZZO SURFACES, AND SINGULAR COMPACTIFICATIONS OF TWO-DIMENSIONAL AFFINE VARIETIES*

DAVID BINDSCHADLER, LAWRENCE BRENTON AND DANIEL DRUCKER

(Received December 5, 1983)

Introduction. Let F be an algebraically closed field and let $(X, \mathcal{O}_X)$ be a two-dimensional, rational, normal compact Gorenstein space over F. If the anticanonical divisor of X is ample, then X is called a (possibly singular) *Del Pezzo surface*. For $F = \boldsymbol{C}$, these surfaces were studied systematically by Du Val [12] in his investigation of the relation between rational double points and subgroups of reflection groups of regular polyhedra (cf. also [8], [13], [20]). They have recently attracted new interest as singular fibres of versal deformations of elliptic singularities ([9], [16], [21], [26]). We have studied certain of these spaces as examples of singular complex surfaces of the homology, cohomology, or homotopy type of $\boldsymbol{CP}^2$ ([2], [7]).

If X is a Del Pezzo surface, the *degree* of X is the integer $d = K \cdot K$, where K is the canonical divisor. If X is singular, each singularity is a double point of type A_k, D_k, or E_k, and the Dynkin diagram Γ corresponding to the singular set is the Coxeter graph of a subgroup of one of the reflection groups A_1, $A_2 + A_1$, A_4, D_5, or E_k, $6 \leqq k \leqq 8$ ([12]). The number n of vertices of Γ is always less than or equal to $9 - d$; if $n = 9 - d$, X will be called *maximally degenerate*. In this case $H^i(X, \boldsymbol{Q}) \cong H^i(\boldsymbol{P}^2, \boldsymbol{Q})$ $\forall i$, with $H^1(X, \boldsymbol{Z}) \cong 0$, $H^2(X, \boldsymbol{Z}) \cong \boldsymbol{Z}$, and $H^3(X, \boldsymbol{Z})$ a finite group of order $\sqrt{(\det(\Gamma))/d}$, where $\det(\Gamma)$ is the determinant of the associated Cartan matrix. Except when X is the singular quadric hypersurface $\boldsymbol{Q}_0^2 \subset \boldsymbol{P}^3(F)$, the Chern class of K generates $H^2(X, \boldsymbol{Z})$ and so the degree also gives the cohomology ring structure. In the maximally degenerate case (but not in general), the singularity type determines the surface up to a deformation through fibres of the same singularity type ([12], [20]).

Now let X be any Del Pezzo surface. Since X is rational, there exists a birational mapping f of X onto $\boldsymbol{P}^2(F)$. Factoring f into a

* Some of the results of Part I of this paper were announced in [3], where a preliminary version of the present paper was referred to under the title: Graph theoretic techniques in algebraic geometry III.