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GENERALIZED INVERSES OF TOEPLITZ OPERATORS AND INVERSE APPROXIMATION IN H^2

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1. Introduction. Let H^2 (resp. H^∞) be the Hardy space of analytic functions in the open unit disc D with square-integrable (resp. essentially bounded measurable) boundary functions, and let π_k ($k \in N := \{0, 1, \cdots\}$) be the linear subspace of all polynomials with degree at most k. Following Chui [1], we then define, for $f \in H^\infty$, the least-squares inverse in π_k of f as the (unique) polynomial $g = g_k$ such that the L^2-norm on the unit circle C

$$\|1 - fg\|_2 := \left\{(2\pi)^{-1}\int_{-\pi}^{\pi} |1 - f(e^{it})g(e^{it})|^2 dt\right\}^{1/2}$$

is minimal when g runs over π_k. Furthermore, the double least-squares inverse $h_{n,k}$ in π_n of f through π_k is defined as the least-squares inverse in π_n of g_k. Using orthogonal polynomials, Chui [1] proved that each g_k is zero-free in the closed unit disc $\bar{D}$, and that if $f \in \pi_n$ then each $h_{n,k}$ is a very good approximant of f in the same π_n which has no zeros in $\bar{D}$.

Now, let A be a (bounded linear) operator on H^2, $\phi \in H^2$ and consider the equation

$$Ag = \phi\,, \quad g \in H^2\,. \tag{1.1}$$

Then an element $g \in H^2$ which minimizes the norm $\|Ag - \phi\|_2$ is called a least-squares solution of (1.1). It is well-known (cf. [3], [7]) that if A has closed range the least-squares solution with minimum norm is unique and is represented as $A^\dagger\phi$, where $A^\dagger$ stands for the (Moore-Penrose) generalized inverse of A. (The generalized inverse is uniquely determined by the four Penrose identities, $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$.)

Suppose that T_f is the Toeplitz operator with symbol $f \in H^\infty$, and that E_k is the orthogonal projection from H^2 onto π_k (as a subspace of H^2). Then the product T_fE_k is of finite rank, and hence has closed range. The solution $(T_fE_k)^\dagger 1 = E_k(T_fE_k)^\dagger 1$ of (1.1) for $A = T_fE_k$, $\phi = 1$ is nothing but the least-squares inverse g_k defined before. Similarly the double least-squares inverse of f is represented as $h_{n,k} = (T_{g_k}E_n)^\dagger 1$. Hence