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LUSIN FUNCTIONS AND NONTANGENTIAL MAXIMAL FUNCTIONS IN THE H^p THEORY ON THE PRODUCT OF UPPER HALF-SPACES

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1. Introduction. In this note, we will give a proof of the L^p norm equivalence between the Lusin area integral $A(u)$ and the nontangential maximal function $N(u)$ of a biharmonic function u defined on the product space $\boldsymbol{D} = \boldsymbol{R}_+^{n_1+1} \times \boldsymbol{R}_+^{n_2+1}$, where $\boldsymbol{R}_+^{n_i+1} = \boldsymbol{R}^{n_i} \times (0, \infty)$ $(i = 1, 2)$.

We will use the following notations. We write

$$(x^{(1)}, y_1; x^{(2)}, y_2) = (x_1^{(1)}, \cdots, x_{n_1}^{(1)}, y_1; x_1^{(2)}, \cdots, x_{n_2}^{(2)}, y_2)$$

for the point of $\boldsymbol{R}^{n_1+1} \times \boldsymbol{R}^{n_2+1}$, where $(x^{(i)}, y_i) \in \boldsymbol{R}^{n_i+1}$, $x^{(i)} = (x_1^{(i)}, \cdots, x_{n_i}^{(i)}) \in \boldsymbol{R}^{n_i}$, and $y_i \in \boldsymbol{R}$ $(i = 1, 2)$. We also write $(x^{(1)}, y_1; x^{(2)}, y_2) = (x, y)$, where $x = (x^{(1)}, x^{(2)}) \in \boldsymbol{R}^N$ $(N = n_1 + n_2)$, and $y = (y_1, y_2) \in \boldsymbol{R}^2$. Let $\boldsymbol{R}_+^{n_i+1} = \{(x^{(i)}, y_i) \in \boldsymbol{R}^{n_i+1}: y_i > 0\}$ $(i = 1, 2)$ and $\boldsymbol{D} = \boldsymbol{R}_+^{n_1+1} \times \boldsymbol{R}_+^{n_2+1}$.

Let $u(x, y)$ be a biharmonic function on $\boldsymbol{D}$, that is, u is twice continuously differentiable and $\Delta_i u = 0$ on $\boldsymbol{D}$ $(i = 1, 2)$, where

$$\Delta_i = \sum_{j=1}^{n_i} (\partial/\partial x_j^{(i)})^2 + (\partial/\partial y_i)^2$$

is the Laplacian in the $(x^{(i)}, y_i)$ variable. For $a = (a_1, a_2)$, $a_1 > 0$, $a_2 > 0$, and $x = (x^{(1)}, x^{(2)}) \in \boldsymbol{R}^N$, we define a product cone $\Gamma_a(x)$ by

$$\Gamma_a(x) = \{(t^{(1)}, y_1; t^{(2)}, y_2) \in \boldsymbol{D}: |t^{(1)} - x^{(1)}| < a_1 y_1, |t^{(2)} - x^{(2)}| < a_2 y_2\} . \tag{1.1}$$

We say that $u \in H^p(\boldsymbol{D})$ $(0 < p < \infty)$ if its nontangential maximal function

$$N_a(u) = \sup\{|u(t, y)|: (t, y) \in \Gamma_a(x)\} \tag{1.2}$$

belongs to the Lebesgue space $L^p(\boldsymbol{R}^N)$. It is known that this definition is independent of a. The Lusin area integral of a biharmonic function u is defined by

$$A_a(u)(x) = \left(\int_{\Gamma_a(x)} |\nabla_1 \nabla_2 u(t, y)|^2 y_1^{1-n_1} y_2^{1-n_2} dt dy\right)^{1/2} , \tag{1.3}$$

where $|\nabla_1 \nabla_2 u|^2 = \sum_{j=1}^{n_1+1} \sum_{k=1}^{n_2+1} |\partial^2/(\partial x_j^{(1)} \partial x_k^{(2)}) u|^2$ with $\partial/\partial x_{n_1+1}^{(1)} = \partial/\partial y_1$, $\partial/\partial x_{n_2+1}^{(2)} = \partial/\partial y_2$. We write $A_{(1,1)}(u) = A(u)$, $N_{(1,1)}(u) = N(u)$, and $\Gamma_{(1,1)}(x) = \Gamma(x)$.