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ON ALGEBRAIC INDEPENDENCE OF SPECIAL VALUES OF GAP SERIES

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Introduction. In this paper we extend the result of Bundschuh and Wylegala [4].

Let $f(z) = \sum_{k=0}^{\infty} a(k)z^{e(k)}$ be a power series, where the $a(k)$ $(k \geqq 0)$ are non-zero algebraic numbers and where the $e(k)$ $(k \geqq 0)$ form an increasing sequence of non-negative integers. We denote by $A(f, n)$ the maximum of the $|\overline{a(k)}|$ $(0 \leqq k \leqq n)$, where for any k $(0 \leqq k \leqq n)$ the $|\overline{a(k)}|$ is the maximum of the absolute values of the conjugates of $a(k)$. We denote by $M(f, n)$ is the least positive integer d such that $d \cdot a(k)$ $(0 \leqq k \leqq n)$ are all algebraic integers, and by $S(f, n)$ is the degree of $\boldsymbol{Q}(a(k); 0 \leqq k \leqq n)$ over $\boldsymbol{Q}$. In [4], Bundschuh and Wylegala proved that $f(\alpha_1), \cdots, f(\alpha_m)$ are algebraically independent for any algebraic numbers $\alpha_1, \cdots, \alpha_m$ with $0 < |\alpha_1| < \cdots < |\alpha_m| < R(f)$, if the condition

$$\lim_{n\to\infty} S(f, n)(e(n) + \log A(f, n) + \log M(f, n))/e(n + 1) = 0$$

is satisfied. In §1, we extend this result as follows. Let $f_i(z) = \sum_{k=0}^{\infty} a(i, k)z^{e(i,k)}$ $(1 \leqq i \leqq m)$ be gap series, where the $a(i, k)$ $(1 \leqq i \leqq m, k \geqq 0)$ are non-zero algebraic numbers and where for any i $(1 \leqq i \leqq m)$ the $e(i, k)$ $(k \geqq 0)$ form an increasing sequence of non-negative integers, and let α_i $(1 \leqq i \leqq m)$ be algebraic numbers with $0 < |\alpha_i| < R(f_i)$. We put $A(n) = \max\{A(f_i, n);\ 1 \leqq i \leqq m\}$, $M(n) = \text{l.c.m.}\{M(f_i, n);\ 1 \leqq i \leqq m\}$, $S(n) = [\boldsymbol{Q}(a(i, k);\ 1 \leqq i \leqq m,\ 0 \leqq k \leqq n): \boldsymbol{Q}]$, $E(n) = \max\{e(i, n), 1 \leqq i \leqq m\}$, $e(n) = \min\{e(i, n);\ 1 \leqq i \leqq m\}$. Then we have the following.

THEOREM. *$f_1(\alpha_1), \cdots, f_m(\alpha_m)$ are algebraically independent if the following two conditions are satisfied*:

(i) $\lim_{n\to\infty} S(n)(E(n) + \log A(n) + \log M(n))/e(n + 1) = 0$;

(ii) $|a(i + 1, n)\alpha_{i+1}^{e(i+1,n)}| = o(|a(i, n)\alpha_i^{e(i,n)}|)$ as $n \to \infty$ $(1 \leqq i \leqq m - 1)$.

Our proof of this result is closely related to the proof of the result of Shiokawa [16].

For example, put $f(z) = \sum_{k=0}^{\infty} z^{k!}$. Let α_j $(1 \leqq j \leqq m)$ be algebraic numbers satisfying $0 < |\alpha_m| < \cdots < |\alpha_1| < 1$. Then the numbers $f^{(i)}(\alpha_j)$ $(0 \leqq i \leqq l,\ 1 \leqq j \leqq m)$ are algebraically independent.