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ON A PROBLEM OF DOOB ABOUT ANGULAR AND FINE CLUSTER VALUES

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Abstract. In 1965, Joseph L. Doob proved that if f is a superharmonic function on a half-space and if $A(p)$ and $F(p)$ denote respectively the angular and fine cluster sets of f at a boundary point p, then one of the following cases holds at almost every boundary point p.

(i) $-\infty \in F(p) \cap A(p)$ and $F(p) \subsetneqq A(p)$.

(ii) f does not have an angular limit at p but has a finite fine limit and an equal normal limit there.

(iii) f has a finite angular limit and an equal fine limit at p.

He then asked whether in (i) the set $F(p)$ can be a proper subinterval of $A(p)$ on a P set of positive measure.

In this note, we study this problem in the two dimensional case. We construct a Nevanlinna's function for which (i) holds for a countably dense set of boundary points. Our result is sharp in the sense that the P set cannot be improved to be of positive measure. It is not clear whether the construction is possible for any P set of measure zero.

1. Introduction. Let H be the right half-plane and let $f(z)$ be a function defined in H. We say that the function f has an angular cluster value v at a boundary point p, if there is a Stolz angle $\Delta(p)$ (i.e., an angle lying in H with one vertex at p) and a sequence $\{p_n\}$ of points in $\Delta(p)$ such that

$$\lim_{n\to\infty} p_n = p \quad \text{and} \quad \lim_{n\to\infty} f(p_n) = v .$$

We shall now follow Brelot [2, p. 327] to introduce the notion of thin set in the sense of Cartan and Brelot. A set E will be said to be ordinarily thin at a point p, if either p is not a limit point of E or there exists a superharmonic function $S(z)$ such that

$$S(p) < \varliminf_{z\to p} S(z) , \qquad \text{where} \quad z \in E - p .$$

The first case is trivial and therefore only the second case will be considered in the sequel.

In contrast to the ordinary thinness, we shall now introduce the

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