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THE SIGNATURE WITH LOCAL COEFFICIENTS OF LOCALLY SYMMETRIC SPACES

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Abstract. We obtain explicit formulae for the (L^2-) signature with local coefficients of certain locally symmetric spaces and then apply them to derive non-vanishing criteria for the middle dimensional cohomology.

Introduction. Let G be a linear connected semisimple Lie group, Γ a discrete subgroup of G and F a finite dimensional complex irreducible G-module. It is well-known (see e.g. [4: VII, §2]) that the Eilenberg-MacLane cohomology space $H^{\cdot}(\Gamma; F)$ is isomorphic to the relative Lie algebra cohomology space $H^{\cdot}(\mathfrak{g}, K; C^{\infty}(\Gamma\backslash G)\otimes F)$, where $\mathfrak{g}$ is the Lie algebra of G and K a maximal compact subgroup of G. If now Γ is co-compact, as we assume for the time being, one also knows that $L^2(\Gamma\backslash G)$ decomposes as a Hilbert direct sum of irreducible subspaces with finite multiplicities,

$$L^2(\Gamma\backslash G) = \sum_{\pi\in\hat{G}}^{\oplus} N_{\Gamma}(\pi)\mathscr{H}_{\pi} ,$$

where $\mathscr{H}_{\pi}$ is the Hilbert space corresponding to $\pi\in\hat{G}$ and $N_{\Gamma}(\pi)$ its multiplicity in $L^2(\Gamma\backslash G)$. The above isomorphism then becomes

$$H^{\cdot}(\Gamma; F) \cong \sum_{\pi\in\hat{G}}^{\oplus} N_{\Gamma}(\pi)H^{\cdot}(\mathfrak{g}, K; \mathscr{H}_{\pi}\otimes F) ,$$

with only finitely many summands giving a non-zero contribution. Specifically (see [4: I, 5.3]), one has

$$H^{\cdot}(\Gamma; F) \cong \sum_{\pi\in\hat{G}_F}^{\oplus} N_{\Gamma}(\pi)H^{\cdot}(\mathfrak{g}, K; \mathscr{H}_{\pi}\otimes F) \tag{0.1}$$

where $\hat{G}_F$ denotes the set of those $\pi\in\hat{G}$ whose infinitesimal character coincides with that of F^*.

Let us now assume that G possesses a non-empty discrete series $\hat{G}_d$. Then $\hat{G}_{d,F} = \hat{G}_d\cap\hat{G}_F$ is also non-empty. Furthermore, if $\pi\in\hat{G}_{d,F}$ then (see [4: II, 5.3]):

$$H^i(\mathfrak{g}, K; \mathscr{H}_{\pi}\otimes F) \cong \begin{cases} 0 , & \text{if } i\neq m \\ \boldsymbol{C} , & \text{if } i = m \end{cases} \tag{0.2}$$

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