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INJECTIVE ENVELOPES OF C^*-DYNAMICAL SYSTEMS*

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Abstract. The injective envelope $I(A)$ of a C^*-algebra A is a unique minimal injective C^*-algebra containing A. As a dynamical system version of the injective envelope of a C^*-algebra we show that for a C^*-dynamical system (A, G, α) with G discrete there is a unique maximal C^*-dynamical system (B, G, β) "containing" (A, G, α) so that $A\times_{\alpha r}G\subset B\times_{\beta r}G\subset I(A\times_{\alpha r}G)$, where $A\times_{\alpha r}G$ is the reduced C^*-crossed product of A by G. As applications we investigate the relationship between the original action α on A and its unique extension $I(\alpha)$ to $I(A)$. In particular, a *-automorphism α of A is quasi-inner in the sense of Kishimoto if and only if $I(\alpha)$ is inner.

1. Introduction. In [10], [12], [13] the author introduced the notion of the *injective envelope* $I(A)$ (resp. *regular monotone completion* $\bar{A}$) of a (not necessarily unital) C^*-algebra A. (Note that a few authors call this $\bar{A}$ the regular completion of A and use the confusing notation $\hat{A}$ instead of $\bar{A}$. But $\hat{A}$ was originally used by Wright [33] to denote the regular σ-completion of A, which is properly contained in $\bar{A}$ in general.) The algebra $I(A)$ is a unique minimal injective C^*-algebra containing A^1 as a C^*-subalgebra with the same unit, where A^1 denotes the C^*-algebra obtained by adjoining a unit to A if A is non-unital and $A\neq\{0\}$, and denotes A itself otherwise. On the other hand, $\bar{A}$ is a unique monotone complete C^*-algebra such that $\bar{A}$ is the monotone closure of A and each $x\in\bar{A}_{sa}$ (the self-adjoint part of $\bar{A}$) is the supremum in $\bar{A}_{sa}$ of the set $\{a\in A^1_{sa}: a\leqq x\}$, where a C^*-algebra B is called *monotone complete* if each bounded increasing net in B_{sa} has a supremum in B_{sa}, and the *monotone closure* of a C^*-subalgebra C of B is the smallest C^*-subalgebra of B containing C which is closed under the formation of suprema in B_{sa} of bounded increasing nets. Moreover, $\bar{A}$ is realized as the monotone closure of A in $I(A)$ and we have canonically $A\subset\bar{A}\subset I(A)$.

The algebra $I(A)$ or $\bar{A}$, being monotone complete AW^*, is more tractable than the original C^*-algebra A and is small enough to inherit some properties of A. For example, $I(A)$ or $\bar{A}$ is an AW^*-factor if and only if A is prime [12, 7.1, 6.3], and if A is unital and simple, then any

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