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AFFINE TORUS EMBEDDINGS WHICH ARE COMPLETE INTERSECTIONS

Dedicated to the memory of Professor Takehiko Miyata

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1. Introduction. Throughout this paper, let k be a field, M a free $\boldsymbol{Z}$-module of finite rank $r \geqq 1$ and N the dual $\mathrm{Hom}(M, \boldsymbol{Z})$ with the canonical pairing $\langle\ ,\ \rangle: M \times N \to \boldsymbol{Z}$. We extend this pairing $\boldsymbol{R}$-linearly to $M_{\boldsymbol{R}} \times N_{\boldsymbol{R}}$ where $M_{\boldsymbol{R}} = \boldsymbol{R} \otimes_{\boldsymbol{Z}} M$ and $N_{\boldsymbol{R}} = \boldsymbol{R} \otimes_{\boldsymbol{Z}} N$. Let σ be a strongly convex rational polyhedral cone in $N_{\boldsymbol{R}}$, i.e., $\sigma = \{\sum_{i=1}^{s} a_i n_i \mid$ any non-negative $a_i \in \boldsymbol{R}\}$ for some $n_i \in N$ $(1 \leqq i \leqq s)$ with $\sigma \cap (-\sigma) = \{0\}$. The dual cone $\sigma^{\vee} = \{x \in M_{\boldsymbol{R}} \mid \langle x, y \rangle \geqq 0$ for all $y \in \sigma\}$ is rational and spans $M_{\boldsymbol{R}}$ as an $\boldsymbol{R}$-vector space. The group algebra $k[M]$ of M over k, whose spectrum T_N is ragarded as a k-split torus, contains the monoid algebra $k[M \cap \sigma^{\vee}]$ of $M \cap \sigma^{\vee}$ over k as a k-subalgebra. Then $\mathrm{Spec}\, k[M \cap \sigma^{\vee}]$, which is denoted by X_σ, is exactly a normal affine equivariant embedding of the torus T_N. Moreover, every normal equivariant embedding of T_N is covered by such X_σ's (e.g., [4, Chap. I]). Consequently some properties on toric singularities should be characterized in terms of convex rational polyhedral cones.

Let us recall the well known hierarchy "regular" $\Rightarrow$ "local complete intersection" $\Rightarrow$ "Gorenstein" $\Rightarrow$ "Cohen-Macaulay" of conditions on X_σ. We already know the following results:

(1.1) (Mumford et al. [4]) *X_σ is nonsingular if and only if σ is nonsingular.*

(1.2) (Ishida [2]) *If $r = 3$ and X_σ is a local complete intersection, then $k[M \cap \sigma^{\vee}]$ is k-isomorphic to $k[x, y, z, w, u]/k[x, y, z, w, u](xz - w^b u^c, yw - u^a)$ for a triple (a, b, c) of non-negative integers.*

(1.3) (Stanley [5]) *$k[M \cap \sigma^{\vee}]$ is a Gorenstein ring if and only if $M \cap \mathrm{int}(\sigma^{\vee}) = m_\sigma + M \cap \sigma^{\vee}$ for an element $m_\sigma \in M$.*

(1.4) (Hochster [1]) *$k[M \cap \sigma^{\vee}]$ is always a Cohen-Macaulay ring.*

Moreover Stanley [6] partially and Watanabe [7] completely classified $M \cap \sigma^{\vee}$ such that X_σ is a local complete intersection under the assumption

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