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ON A MOMENT PROBLEM

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Abstract. Let n_0 be any fixed non-negative integer, $-\infty \leqq a < b \leqq \infty$ and $f(x) \geqq 0$ an absolutely continuous function with $f'(x) \neq 0$, a.e. on (a, b). Then the sequence of functions $\{(f(x))^n e^{-f(x)}\}_{n=n_0}^{\infty}$ is complete in $L(a, b)$ if and only if the function $f(x)$ is strictly monotone on (a, b).

1. Introduction. Let $-\infty \leqq a < b \leqq \infty$, and let $L(a, b)$ be the space of all summable functions defined on the interval (a, b). Then a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ is said to be complete in $L(a, b)$ if for every $g \in L(a, b)$, the equalities

$$\int_a^b g(x) f_n(x) dx = 0 , \qquad \text{for all} \quad n = 1, 2, \cdots ,$$

imply $g(x) = 0$, a.e. (almost everywhere) on (a, b). The well-known Müntz-Szász theorem (Boas [1, p. 235]) is concerned with a complete sequence of functions in $L(a, b)$, where (a, b) is a *bounded* interval, and is stated as follows:

THEOREM A. *Let $0 \leqq a < b < \infty$ and $0 < n_1 < n_2 < \cdots$. Then $\{x^{n_i}\}_{i=1}^{\infty}$ is complete in $L(a, b)$ if and only if $\sum_{i=1}^{\infty} 1/n_i = \infty$.*

In this paper, we shall first consider the completeness of a sequence of functions $\{x^n e^{-x}\}$ in $L(a, \infty)$, where $a \geqq 0$ (Theorem 1), then use a theorem of Zarecki to extend the result just obtained to the sequence of functions $\{(f(x))^n e^{-f(x)}\}$ (Theorem 2). Finally, we give some remarks on Laguerre and Hermite functions.

THEOREM 1. *For any fixed integer $n_0 \geqq 0$ and for any fixed real number $a \geqq 0$, the sequence of functions $\{x^n e^{-x}\}_{n=n_0}^{\infty}$ is complete in $L(a, \infty)$.*

THEOREM 2. *Let n_0 be any fixed non-negative integer, $-\infty \leqq a < b \leqq \infty$ and $f(x) \geqq 0$ an absolutely continuous function with $f'(x) \neq 0$, a.e. on (a, b). Then the sequence of functions $\{(f(x))^n e^{-f(x)}\}_{n=n_0}^{\infty}$ is complete in*

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