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STRONG AND CLASSICAL SOLUTIONS OF THE HOPF EQUATION —AN EXAMPLE OF FUNCTIONAL DERIVATIVE EQUATION OF SECOND ORDER

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Abstract. In this note, we construct, strong and classical solutions of the Hopf equation, a statistical version of the Navier-Stokes equation on a compact Riemannian manifold with or without boundary. Our points are to regard the Hopf equation as a given Functional Derivative Equation (F.D.E. for short) of second order, to derive the Navier-Stokes equation as the characteristic equation of it and to give an exact meaning to the 'trace' of the second order functional derivatives which appear in the Hopf equation. To construct a solution of the Hopf-Foiaş equation with the energy inequality of strong form, we apply Foiaş's argument with slight modifications instead of using Prokhorov's compactness argument.

1. Introduction. Let (M, g) be a compact Riemannian manifold of dimension d with or without boundary ∂M. We denote by $\mathring{\boldsymbol{X}}_\sigma(M)$ and $\mathring{\Lambda}^1_\sigma(M)$, the space of all solenoidal vector fields on M which vanish near the boundary and that of all divergence free 1-forms on M which vanish near the boundary, respectively. $\tilde{\boldsymbol{H}}$ (resp. $\boldsymbol{H}$) stands for the completion of the space $\mathring{\Lambda}^1_\sigma(M)$ (resp. $\mathring{\boldsymbol{X}}_\sigma(M)$) with respect to $\tilde{\boldsymbol{L}}^2$-norm (resp. $\boldsymbol{L}^2$-norm).

The aim of this paper is to solve the following problem.

(I) Find a real functional $W(t, \eta)$ on $[0, \infty)\times\tilde{\boldsymbol{H}}$, satisfying

$$\text{(I.1)}\qquad \frac{\partial}{\partial t}W(t, \eta) = \int_M \left[-i\left\{\frac{\partial}{\partial x^k}\eta_j(x) - \Gamma^l_{jk}(x)\eta_l(x)\right\}\frac{\delta^2 W(t, \eta)}{\delta\eta_j(x)\delta\eta_k(x)} + \nu(\Delta\eta)_j(x)\frac{\delta W(t, \eta)}{\delta\eta_j(x)} + i\eta_j(x)f^j(x, t)W(t, \eta)\right]d_g x\ ,$$

$$\text{(I.2)}\qquad \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^j}\left\{\sqrt{g(x)}\frac{\delta W(t, \eta)}{\delta\eta_j(x)}\right\} = 0\ ,$$

for $\eta = \eta(x) = \eta_j(x)dx^j \in \mathring{\Lambda}^1_\sigma(M)$ and $t \in (0, \infty)$, and

$$\text{(I.3)}\qquad W(t, 0) = 1\ ,$$

$$\text{(I.4)}\qquad W(0, \eta) = W_0(\eta)\ .$$

Here $f(x, t) = f^j(x, t)\partial/\partial x^j \in \mathring{\boldsymbol{X}}_\sigma(M)$ for each t and $W_0(\eta)$ is a given positive definite functional on $\tilde{\boldsymbol{H}}$ satisfying