Tôhoku Math. Journ.
39 (1987), 105–114.

A FUNDAMENTAL VARIATIONAL LEMMA FOR EXTREMAL QUASICONFORMAL MAPPINGS COMPATIBLE WITH A FUCHSIAN GROUP

KEN-ICHI SAKAN

(Received February 18, 1986)

1. Introduction. Let Γ be a Fuchsian group acting on the upper half-plane U. A holomorphic function ϕ on U is called a quadratic differential for Γ on U if $\phi(\gamma(z))\gamma'(z)^2 = \phi(z)$ for every $\gamma \in \Gamma$. Let σ be a Γ-invariant closed subset of the extended real line $\hat{R} = \partial U$, which contains 0, 1 and ∞ and satisfies $A(\Gamma, \sigma) \neq \{0\}$. Here $A(\Gamma, \sigma)$ denotes the space consisting of all the quadratic differentials ϕ for Γ on U, which are continuously extensible to $\hat{R} \setminus \sigma$ and real on $\hat{R} \setminus \sigma$, and satisfy

$$\|\phi\|_{U/\Gamma} \equiv \iint_{U/\Gamma} |\phi(z)| dxdy < \infty .$$

Let E be a Γ-invariant measurable, possibly empty, subset of U, where $U \setminus E$ has positive area measure and where, if $\dim A(\Gamma, \sigma) = \infty$, then E/Γ is assumed to be relatively compact in $\{U \cup (\hat{R} \setminus \sigma)\}/\Gamma$. In what follows, such a configuration (Γ, σ, E) is said to be admissible.

The "fundamental variational lemma", referred to in the title, is stated as Theorem in Section 4, and is formulated for an arbitrary admissible configuration (Γ', σ', E'). It is a generalized form of the corresponding one in Reich [7]. Quite recently, in [2], Fehlmann has succeeded in proving it in the case $\Gamma' = 1$.

In this note, following the method in [2] with some parts modified, we give the proof of our Theorem. In other words, we prove the fundamental variational lemma for an arbitrary admissible configuration (Γ', σ', E'). In [10], Corollary to our Theorem will be applied to the characterization problem of extremal quasiconformal (qc, for short) mappings compatible with an arbitrary Fuchsian group with an arbitrary dilatation bound. To be more specific, in [10], Corollary to our Theorem will play an important role in characterizing extremal qc mappings within a class $Q \equiv Q(\Gamma, h, \sigma, E, b)$ when Γ', σ' and E' are suitably chosen for such a given class Q (see Section 2 for the definitions of h, b and Q). This is the reason why we formulate our Theorem and Corollary by using notation (Γ', σ', E'), instead of (Γ, σ, E), for an admissible configuration.