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THE RANGE AND PSEUDO-INVERSE OF A PRODUCT

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Abstract. By definition the cosine of the angle between the two subspaces M and N is $\sup\{|\langle u, v\rangle| : u \in M,\ v \in N,\ \|u\| = 1 = \|v\|\}$. For operators A and B with closed range in Hilbert spaces, AB has closed range if and only if the angle between $\ker A$ and $B((\ker AB)^{\perp})$ is positive. Moreover, if we denote by $A^{\dagger}$ the pseudo-inverse of A, then $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if $B((\ker AB)^{\perp}) \subset (\ker A)^{\perp}$ and $A^*((\ker B^*A^*)^{\perp}) \subset (\ker B^*)^{\perp}$.

Let H, K, L be Hilbert spaces over complex field. For a subspace $M \subset H$, we denote by $M^{\perp}$ the orthogonal complement of M and by $\bar{M}$ the closure of M. For two subspaces M and N of H, we shall say that the angle between M and N is θ, if

$$\cos\theta = \sup\{|\langle u, v\rangle| : u \in M,\ v \in N,\ \|u\| = 1 = \|v\|\} .$$

For convenience, we denote the angle by $\theta(M, N)$. Let $C(H, K)$ (resp. $B(H, K)$) be the set of all closed linear operators (resp. bounded operators) from H to K. For $T \in C(H, K)$ we denote the domain of T by $D(T)$, the kernel of T by $\ker T$ and the range of T by $R(T)$. Each $T \in C(H, K)$ induces a one-to-one operator from $(\ker T)^{\perp}$ onto TH. This induced operator is invertible. Define $T^{\dagger}$ to be that inverse on TH and to be zero on $(TH)^{\perp}$. We call $T^{\dagger}$ the pseudo-inverse of T. $T^{\dagger}$ is bounded if and only if $R(T)$ is closed (cf. [3, Theorem 3.1.2]). If $H = K$, we write $B(H)$ instead of $B(H, H)$.

A basic problem in the theory of pseudo-inverse is to determine when the range of a product is closed and the pseudo-inverse of a product is the product of the pseudo-inverses. For $A, B \in B(H)$ with closed range, Bouldin [1] indicated that the simple geometric condition

$$\theta(\ker A \cap (\ker A \cap BH)^{\perp}, BH) > 0 \tag{1}$$

is both necessary and sufficient for AB to have closed range. Furthermore, he proved in [2] that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if the following conditions hold:

$$(AB)^{\dagger} \text{ is bounded ;} \tag{2}$$

$$A^*H \text{ is invariant under } BB^* \text{ ;} \tag{3}$$

$$A^*H \cap \ker B^* \text{ is invariant under } A^*A \text{ .} \tag{4}$$