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LUSIN FUNCTIONS ON PRODUCT SPACES

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1. Introduction. In [1] and [2], Calderón and Torchinsky introduced the parabolic H^p spaces associated with a group of linear transformations of $\boldsymbol{R}^d$ and obtained analogues of some results of Fefferman-Stein [8] in this context. Later Gundy-Stein [11] extended some of the results of [8] to the product spaces. (See also Gundy [10], M. P. and P. Malliavin [13].) On the other hand, it seems likely that some parts of the theory of Calderón-Torchinsky [1], [2] also extend to the product spaces. In fact, in the present note we prove the equivalence with respect to the L^p-"norms" of the Lusin functions and the nontangential maximal functions arising from certain two-parameter families of linear transformations of $\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}$ (see Theorem 1 and the corollary in § 3), which is an extension to the product spaces of a special case of a result of [1] and also is a generalization of a result of Gundy-Stein [11]. Combined with the argument of Fefferman-Stein [9], this enables us to extend Fefferman's weak type estimates (see [7]) to the case of the double singular integrals with mixed homogeneity (see Theorem 3 in § 3).

2. Preliminaries.

2.1. Let $x\in\boldsymbol{R}^n$ $(n\geqq 2)$. We write $x=(x^{(1)}, x^{(2)})$, where $x^{(1)}\in\boldsymbol{R}^{n_1}$, $x^{(2)}\in\boldsymbol{R}^{n_2}$ $(n_1, n_2\geqq 1,\ n_1+n_2=n)$ and $x^{(i)}=(x_1^{(i)},\cdots,x_{n_i}^{(i)})$ $(i=1,2)$. If $X\in\boldsymbol{R}^{n_1+1}\times\boldsymbol{R}^{n_2+1}$, we write $X=(x^{(1)}, t_1; x^{(2)}, t_2)$; $x^{(i)}\in\boldsymbol{R}^{n_i}$, $t_i\in\boldsymbol{R}$. (We often write, for example, "$x^{(i)}\in\boldsymbol{R}^{n_i}$" instead of "$x^{(1)}\in\boldsymbol{R}^{n_1}$ and $x^{(2)}\in\boldsymbol{R}^{n_2}$" for simplicity. This abbreviation will be used throughout.) We also write $(x^{(1)}, t_1; x^{(2)}, t_2)=(x, t)$, where $x=(x^{(1)}, x^{(2)})$, $t=(t_1, t_2)$.

Set $\boldsymbol{R}_+^{n_i+1}=\{(x^{(i)}, t_i)\in\boldsymbol{R}^{n_i+1}: t_i>0\}$ $(i=1,2)$ and $\boldsymbol{D}=\boldsymbol{R}_+^{n_1+1}\times\boldsymbol{R}_+^{n_2+1}$.

2.2. Let P_i be a linear transformation of $\boldsymbol{R}^{n_i}$ such that $(P_ix^{(i)}, x^{(i)})\geqq (x^{(i)}, x^{(i)})$ for all $x^{(i)}\in\boldsymbol{R}^{n_i}$, where $(x^{(i)}, y^{(i)})$ denotes the ordinary inner product in $\boldsymbol{R}^{n_i}$. We consider a group $A_{t_i}^{(i)}=t_i^{P_i}$ $(0<t_i<\infty)$ of linear transformations of $\boldsymbol{R}^{n_i}$.

For $x^{(i)}\in\boldsymbol{R}^{n_i}-\{0\}$, let us denote by $\rho^{(i)}(x^{(i)})$ the unique t_i such that

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