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SPECIAL DIVISORS AND VECTOR BUNDLES

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Introduction. Let X be a nonsingular, complete curve of genus $g \geqq 3$ over $\boldsymbol{C}$, the field of complex numbers and let J be the Jacobian of X, the space of isomorphism classes of line bundles of degree 0 on X. It is a complex torus of dimension g. If we denote by ϕ the Abel-Jacobi mapping of X_d, the d-fold symmetric product of the curve, into J, Abel's theorem assures us that $\phi^{-1}(\phi(D))$ is nothing but the projective linear system $P(H^0(L_D))$, associated to the line bundle L_D given by the effective divisor $D \in X_d$ on X. When d lies between 0 and $(g-1)$, $\phi(X_d)$ is a proper subvariety of J. This subvariety $\phi(X_d)$ admits a natural filtration by subvarieties

$$\phi(X_d) = W_d^0 \supseteqq W_d^1 \cdots \supseteqq W_d^r \cdots$$

defined in terms of the dimension of the fibre of ϕ. For example, $W_d^r = \{\phi(D) \mid \text{dimension of } \phi^{-1}(\phi(D)) \geqq r\}$. It is a classical problem to study the structure of these special linear systems.

Formally, one may define an effective divisor D on X to be special if $H^1(X, L_D) \neq 0$.

In 1874, A. Brill and M. Noether published their investigations on special linear systems and conjectured that on a very general curve X, the dimension of W_d^r is given by $\rho(r, d) = g - (r+1)(g-d+r)$. In 1980, Griffiths and Harris [G-H] settled this conjecture affirmatively.

Picking up the thread from here, we extend the notion of special divisors to stable vector bundles on X. Indeed, a vector bundle V on X is said to be stable, if for every subbundle $W \subsetneqq V$ with $W \neq 0$, $\mu(W) = (\text{degree } W)/(\text{rank } W) < \mu(V)$. Such a bundle with *nonnegative degree* is said to be *special if* $H^1(X, V) \neq 0$. Replacing J, the isomorphism classes of line bundles of degree zero, by $U_{n,d}$, the variety consisting of isomorphism classes of stable bundles of rank n and degree d we may define

$$W_d^r = \bar{U}_d^r\,, \quad U_d^r = \{V \in U_{n,d} \mid h^0(X, V) \geqq r+1\}$$

where W_d^r is the Zariski closure of U_d^r in $M_{n,d}$, the "natural compactification" of $U_{n,d}$. (See Section 1.12.2, Chapter I).

In this article, we undertake investigations of