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COMPACT OPERATORS IN TYPE III$_\lambda$ AND TYPE III$_0$ FACTORS, II

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1. Introduction and notations. In this paper we continue the program started in [8] of studing notions of compact operators in type III$_\lambda$ $(0 \leqq \lambda < 1)$ factors. Given a type III$_\lambda$ factor $\boldsymbol{M}$ operating on a separable Hilbert space $\boldsymbol{H}$, we represent it as the crossed product of a type II$_\infty$ algebra $\boldsymbol{N}$ (a factor for $0<\lambda<1$ or an algebra with diffuse center for $\lambda=0$) by an automorphism θ that λ-scales a trace τ (i.e., $\tau\circ\theta = \lambda\tau$ for $0<\lambda<1$ or $\tau\circ\theta \leqq \lambda_0\tau$ with $\lambda_0<1$ for $\lambda=0$). We embed $\boldsymbol{N}$ in $\boldsymbol{M}$ and let E be the canonical normal conditional expectation $E\colon \boldsymbol{M}\to\boldsymbol{N}$, u be the unitary operator implementing θ (i.e., $\mathrm{Ad}\, u = \theta$) such that $\{\boldsymbol{N}, u\}'' = \boldsymbol{M}$ and $\varphi = \tau\circ E$ be the dual weight of τ. Then φ is a lacunary weight, i.e., 1 is an isolated point in $\mathrm{Sp}\,\sigma^\varphi$, $\lambda_0 = \sup\{\lambda\in \mathrm{Sp}\,\sigma^\varphi \mid \lambda<1\}$, $\boldsymbol{N}$ is the centralizer of φ and $\boldsymbol{M}\cap\boldsymbol{N}' = \boldsymbol{N}\cap\boldsymbol{N}'$. For further references see [2, § 4, 5] and [16, § 30.4].

In [8] we denoted by $\boldsymbol{I}(\boldsymbol{N})$ the two sided ideal of $\boldsymbol{N}$ generated by the finite projections of $\boldsymbol{N}$, by $\boldsymbol{J}(\boldsymbol{N})$ the norm closure of $\boldsymbol{I}(\boldsymbol{N})$ and we defined

$$\boldsymbol{I} = \mathrm{span}\{x\in\boldsymbol{M}^+ \mid E(x)\in\boldsymbol{I}(\boldsymbol{N})\}\,,$$
$$\boldsymbol{M}_\varphi = \mathrm{span}\{x\in\boldsymbol{M}^+ \mid \varphi(x)<\infty\}\,,$$
$$\boldsymbol{J} = \bar{\boldsymbol{I}} \quad \text{where the bar denotes the norm closure}\,.$$

We then obtained the embeddings for $0<\lambda<1$ [8, Theorem 6.2]

$$\boldsymbol{I}\subset\boldsymbol{M}_\varphi\subset\boldsymbol{J}$$

analogous to the classical embeddings of finite rank, trace-class and compact operator ideals. For the case $\lambda=0$ we obtained a similar embedding involving the center of $\boldsymbol{N}$ [8, Corollary 6.5]. We then proved the generalization of several of the classical properties of compact operators, (Riesz, Calkin, Rellich and Hilbert properties [8, Theorem 5.3, Proposition 5.6]). In [8, Remark 4.6] we noticed that $\boldsymbol{J}$ is minimal among the C^*-algebras $\boldsymbol{C}\supset\boldsymbol{E}(\boldsymbol{C}) = \boldsymbol{J}(\boldsymbol{N})$ which are two sided $\boldsymbol{N}$-modules, while the maximal one is the algebra $\boldsymbol{K}$ given by:

DEFINITION 1.1. $\boldsymbol{K} = \mathrm{span}\{x\in\boldsymbol{M}^+ \mid E(x)\in\boldsymbol{J}(\boldsymbol{N})\}$.