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AN ANALOGUE OF THE HOLONOMY BUNDLE FOR A FOLIATED MANIFOLD

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1. Introduction. Let $(M, \mathcal{F})$ be a smooth foliated manifold with M connected. Let $E \subset T(M)$ be the tangent bundle of $\mathcal{F}$, and let $D \subset T(M)$ be a subbundle satisfying $T(M) = E \oplus D$.

A horizontal curve is a piecewise smooth curve $\sigma: [0, 1] \to M$ whose tangent vector field lies in D. For $x \in M$, let $P(x)$ be the set of points in M that can be joined to x by a horizontal curve. Clearly the sets $P(x)$ partition M. The purpose of this paper is to investigate the structure of these sets. We show that under certain geometric conditions the sets $P(x)$ are immersed submanifolds of M.

As a special case we consider the situation in which M is a Riemannian manifold, and D is the distribution orthogonal to the leaves. We show that the geometric conditions implying that the sets $P(x)$ are immersed submanifolds are satisfied in the following cases.

(1) $\mathcal{F}$ is totally geodesic and the induced metrics on the leaves are complete (cf. [2]).

(2) $\mathcal{F}$ is totally umbilic with $\dim(\mathcal{F}) \geqq 3$, and the induced conformal structures on the leaves are complete.

(3) $\mathcal{F}$ is totally umbilic with $\dim(\mathcal{F}) \geqq 3$ and the metric on M is complete and bundle-like.

(4) The second fundamental form of the leaves is Bott parallel and the metric on M is complete and bundle-like.

(5) A certain tensor defined in terms of the second fundamental form of the leaves and the Bott connection vanishes, and the induced metrics on the leaves are complete.

(6) The above tensor has a specific form, $\dim(\mathcal{F}) \geqq 2$, and the induced projective structures on the leaves are complete.

REMARK. If $M \to N$ is a principal bundle, $\mathcal{F}$ is the foliation of M by the fibers, and D is a connection in M, then the sets $P(x)$ are just the holonomy bundles of the connection.

2. Definitions. For each horizontal curve $\sigma: [0, 1] \to M$ there exists a family of diffeomorphisms $\phi_t: V_0 \to V_t$ $(0 \leqq t \leqq 1)$ such that