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ROOT STRINGS WITH THREE OR FOUR REAL ROOTS IN KAC-MOODY ROOT SYSTEMS

Dedicated to Professor Eiichi Abe on his sixtieth birthday

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0. Introduction. A characterization and a presentation of a (universal) Kac-Moody group over a field (of any characteristic) have been given by Tits [6]. Such a presentation, which is a natural generalization of Steinberg's one for a (simply connected) split semisimple algebraic group over a field (cf. [5]), is conjectured by E. Abe and established by J. Tits. The most interesting part of the presentation is the so-called "commutation relation", which is deeply related to the root strings and whose explicit description is given in [4]. In this paper, we will discuss certain root strings in Kac-Moody root systems, and give some direct applications to the associated Kac-Moody groups. Our main result is as follows.

Let $A=(a_{ij})$ be an $n\times n$ generalized Cartan matrix, Δ the associated root system, and Δ^{re} the set of real roots. Put $r(\alpha;\beta)=\sharp|\{\beta+k\alpha\,|\,k\in\boldsymbol{Z}\}\cap\Delta^{\mathrm{re}}|$ for $(\alpha,\beta)\in\Delta^{\mathrm{re}}\times\Delta$. Then the following two conditions are equivalent.

(1) $r(\alpha;\beta)=3$ or 4 for some $(\alpha,\beta)\in\Delta^{\mathrm{re}}\times\Delta$.

(2) $a_{ij}=-1$ and $a_{ji}<-1$ for some i, j $(1\leqq i, j\leqq n)$.

As a corollary, we can simplify the Steinberg-Tits presentation of the associated Kac-Moody group in the case when A has a certain property.

1. Notation and lemmas. Let $A=(a_{ij})_{i,j\in I}$ be an $n\times n$ generalized Cartan matrix, $(\mathfrak{h},\Pi,\Pi^{\vee})$ a realization of A, and $\mathfrak{g}(A)$ the Kac-Moody Lie algebra (over $\boldsymbol{C}$ associated with A), where $I=\{1,2,\cdots,n\}$, $\Pi=\{\alpha_1,\cdots,\alpha_n\}$, $\Pi^{\vee}=\{h_1,\cdots,h_n\}$ and $\alpha_i(h_j)=a_{ji}$ (cf. [1]). We denote by W the Weyl group with simple reflections $w_1,\cdots,w_n$. Let Δ be the root system of $\mathfrak{g}(A)$ with Π as simple roots, $\Delta^{\mathrm{re}}=\{w(\alpha)\,|\,w\in W,\alpha\in\Pi\}$ the set of real roots, Δ_+ the set of positive roots, and Δ_+^{re} the set of positive real roots. For each $\alpha\in\Delta^{\mathrm{re}}$, let $h_\alpha\in\mathfrak{h}$ be the dual root of α. Then both $\alpha(h_\beta)$ and $\beta(h_\alpha)$ have the same sign (one of $+$, 0, $-$) for all $\alpha,\beta\in\Delta^{\mathrm{re}}$ (cf. [3]). Put $\mathrm{ht}(\alpha)=\sum_{k=1}^{n}c_k$, called the height of α, if $\alpha=\sum_{k=1}^{n}c_k\alpha_k\in\Delta$. Let $S(\alpha;\beta)=\{\beta+k\alpha\,|\,k\in\boldsymbol{Z}\}\cap\Delta$ for $(\alpha,\beta)\in\Delta^{\mathrm{re}}\times\Delta$. This $S(\alpha;\beta)$ is called