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LINEAR DIFFERENTIAL EQUATIONS MODELED AFTER HYPERQUADRICS

Dedicated to Professor Ichiro Satake on his sixtieth birthday

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0. Introduction. In this paper, we study systems of linear partial differential equations in n ($\geqq 3$) variables of rank (=the dimension of the solution space) $n+2$. The case $n=2$ is treated in [SY1] and [SY2].

Here we would like to mention our motivation. Let D be the symmetric domain of type IV of dimension n ($\geqq 3$), Γ be a transformation group acting properly discontinuously on D, X be a quotient variety of D under Γ naturally equipped with the structure of orbifold, π be the projection of D onto X and finally let φ be the inverse map $\pi^{-1}: X \to D$, which is called the *developing map of the orbifold X*. We think there should be a system of linear differential equations (E) defined on X such that the solution of the system gives rise to the map φ. It is called the *uniformizing differential equation of the orbifold X*. Since D can be thought of as a part of a non-degenerate quadric hypersurface Q in P^{n+1} and since we have the following inclusion relations

$$\mathrm{Aut}(D) \subset \mathrm{Aut}(Q) \subset \mathrm{Aut}(P^{n+1}) \cong PGL(n+2)$$

of the groups of complex analytic automorphisms, the system (E) must be of rank $n+2$ and the mapping defined on X by the ratio of $n+2$ linearly independent solutions of (E) has its image in the hyperquadric Q. In this way we encounter equations in n variables of rank $n+2$. Making a linear change of independent variables $x=(x^1, \cdots, x^n)$ if necessary, we may assume that any system in n variables of rank $n+2$ with the unknown w has the form

$$\text{(EQ)} \qquad \frac{\partial^2 w}{\partial x^i \partial x^j} = g_{ij} \frac{\partial^2 w}{\partial x^1 \partial x^n} + \sum_{k=1}^n A_{ij}^k \frac{\partial w}{\partial x^k} + A_{ij}^0 w \qquad (1 \leqq i, j \leqq n)$$

where

$$g_{ij} = g_{ji},\ A_{ij}^k = A_{ji}^k,\ A_{ij}^0 = A_{ji}^0,\ g_{1n} = 1,\ A_{1n}^k = A_{1n}^0 = 0\,.$$

This system is the key to connecting the theory of conformal connections, the projective

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