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HARMONIC INNER AUTOMORPHISMS OF COMPACT CONNECTED SEMISIMPLE LIE GROUPS

JOON-SIK PARK

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0. Introduction. Harmonic maps of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) are the extrema of the energy functional (cf. [1])

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2 dV_g .$$

In this paper, we treat the case $(M, g)=(N, h)=(G, g)$ for a compact connected semisimple Lie group G with a left invariant Riemannian metric g. It is well known that every inner automorphism of G into itself is both isometric and harmonic with respect to a bi-invariant Riemannian metric g_0 on G. However, we here deal with an arbitrary left invariant metric g on G, and show which inner automorphisms of G into itself are harmonic maps of (G, g) into itself.

In §1, we introduce Guest's criterion (cf. Lemma A) for the map between reductive homogeneous spaces G/H and G'/H' induced by a Lie group homomorphism from G into G'.

In §2, using this criterion, we obtain a necessary and sufficient condition for an inner automorphism A_x of (G, g) to be harmonic (cf. Theorem 2.2).

In the particular case $G=SU(2)$, we then completely determine harmonic inner automorphisms of $(SU(2), g)$ for every left invariant Riemannian metric g (cf. Proposition 3.3–3.5).

Finally in Theorems 3.6 and 3.7, we show that for any left invariant and but not bi-invariant Riemannian metric g on $G=SU(2)$, there always exist on (G, g) both a non-harmonic inner automorphism and a non-isometric but harmonic inner automorphism.

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1. Preliminaries. In this section, we review Guest's work which gives a necessary and sufficient condition for the map induced by a homomorphism $\theta: G\to G'$ between reductive homogeneous spaces G/H, G'/H' with invariant Riemannian metrics to be