A TRANSPLANTATION THEOREM FOR
LAGUERRE SERIES

YUICHI KANJIN

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1. Introduction. Let $L_n^\alpha(x)$, $\alpha > -1$, be the Laguerre polynomial of degree $n$ and of order $\alpha$ defined by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \left( \frac{d}{dx} \right)^n \left( e^{-x} x^{\alpha+n} \right).$$

Then the functions $\tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2}$, $n = 0, 1, 2, \ldots$, are orthonormal on the interval $(0, \infty)$ with respect to the ordinary Lebesgue measure $dx$, where

$$(\tau_n^\alpha)^{-2} = \int_0^\infty \{ L_n^\alpha(x) \}^2 e^{-x} x^\alpha dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}.$$ 

This orthonormal system leads us to the formal expansion of a function $f(x)$ on $(0, \infty)$:

$$f(x) \sim \sum_{n=0}^{\infty} a_n^\alpha(f) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2},$$

where $a_n^\alpha(f)$ is the $n$-th Fourier-Laguerre coefficient of order $\alpha$ of $f(x)$ defined by

$$a_n^\alpha(f) = \int_0^\infty f(x) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} dx.$$ 

We note that the integral converges and $a_n^\alpha(f)$ is finite if $\alpha \geq 0$ and $1 \leq p \leq \infty$, or if $-1 < \alpha < 0$ and $(1 + \alpha/2)^{-1} < p \leq \infty$.

Our theorem is as follows:

THEOREM. Let $\alpha, \beta > -1$ and $\gamma = \min \{ \alpha, \beta \}$. If $\gamma \geq 0$, then

$$\int_0^\infty \left\{ \sum_{n=0}^{\infty} a_n^\beta(f) \tau_n^\beta L_n^\beta(x) e^{-x/2} x^{\alpha/2} \right\}^p dx \leq C \int_0^\infty |f(x)|^p dx$$

for $1 < p < \infty$, where $C$ is a constant independent of $f$. If $-1 < \gamma < 0$, then (1.1) holds for $(1 + \gamma/2)^{-1} < p < -2/\gamma$.