AN ALGEBRAIC HALFWAY MODEL FOR THE EVERSION OF THE SPHERE

(WITH AN APPENDIX BY BERNARD MORIN)

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Abstract. In this paper, I construct a new version of the halfway model for the eversion of the sphere, called the closed halfway model, whose image can readily be shown to be the set of zeros of an explicit polynomial of degree eight. For this purpose, a 4-parameter family of halfway models is thoroughly investigated. This family also contains the so-called open halfway model constructed in [A2]. The closed halfway model is chosen among the immersions of this family whose multiple loci contain two circles. Applied to the results of [A1], a similar study leads to notice that there exist Boy surfaces depending on two parameters, each of which intersects a given sphere along four circles (one parallel and three meridians). In the Appendix, Morin gives a coding in differential topological terms, of a sphere eversion which turns out to be minimal in many respects, so that, from now on, we no longer need to refer to pictures in order to present the subject.

Introduction. The present paper is the first step in the program we would eventually like to carry out. Indeed, the task we have in mind, is to construct an eversion of the sphere in terms of a continuous family of immersions of \( S^2 \) into \( \mathbb{R}^3 \) such that the images of all members of the family be real algebraic surfaces, i.e., the sets of zeros of some polynomials in three variables. Obviously, we are looking for surfaces with minimal complexity (where the word complexity must be understood in a rather vague sense), so that the singular locus of each surface could be controlled in such a way that computations should lead to an easy description of such a homotopy. Assuming that the eversion has some symmetry with respect to time, the first task, in order to solve the problem, is to build a handy central step for the eversion. Here we present an algebraic candidate for this so-called halfway model which turns out to be a good one since it minimizes the complexity in many respects.

In fact, there are two differential types of the halfway model, the open halfway model and the closed halfway model, as B. Morin called them. First, we obtain and examine an algebraic version of the open halfway model, already mentioned in [A2], by modifying the construction given in [A1] for the Boy surface, in such a way that this threefolded symmetric object is now replaced by a fourfold surface. Instead of the

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