Tôhoku Math. J.
44 (1992), 251–258

YAMABE METRICS AND CONFORMAL TRANSFORMATIONS

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(Received April 11, 1991)

Abstract. We derive higher order variational formulas for the Yamabe functional, and give an example of infinitesimal deformation of a solution of the Yamabe problem which does not come from conformal vector field.

The Yamabe theorem, which was proved by Schoen [7], states that for any conformal class on a compact cnnected manifold there exists a metric of constant scalar curvature which minimizes the Yamabe functional (see §1) defined on the conformal class. In this paper, we are interested in the space of solutions of the Yamabe problem, that is, the space of minimizers for the Yamabe functional. The conformal transformation group acts naturally on this space, and a naïve question will be whether this action is transitive (up to homothety) or not. We shall show new necessary conditions for a vector field to be conformal, and give examples which negatively answer the question at the infinitesimal level.

1. The space of Yamabe metrics. Let M be a compact connected n-manifold, and C a conformal class of Riemannian metrics of M, i.e., $C=\{e^{2u}g;\ u\in C^{\infty}(M)\}$ for any fixed metric $g\in C$. Throughout this paper, we assume that the dimension n is at least 3. The *Yamabe functional* $I\colon C\to \boldsymbol{R}$ is defined as

$$I(g)=\int_M R_g dv_g \Big/ \left(\int_M dv_g\right)^{(n-2)/n} \qquad \text{for} \quad g\in C\,,$$

where R_g is the scalar curvature function of a metric $g\in C$. We set

$$S(M, C)=\{g\in C;\ I(g)=\mu(M, C)\}\,,$$

where

$$\mu(M, C)=\inf\{I(g);\ g\in C\}\,.$$

We call a metric in $S(M, C)$ a solution of the Yamabe problem, or simply a *Yamabe metric*. Since a Yamabe metric is a minimizer of $I\colon C\to \boldsymbol{R}$, variational formulas show the following properties for $g\in S(M, C)$:

$$R_g=\text{const.} \tag{1.1}$$

1991 *Mathematics Subject Classification*. Primary 53C20.