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ON SOAP BUBBLES AND ISOPERIMETRIC REGIONS IN NON-COMPACT SYMMETRIC SPACES, I

Dedicated to Professor Ichiro Satake

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1. Introduction. In a given Riemannian manifold M^n, a *soap bubble* is a closed hypersurface of constant mean curvature; an *isoperimetric region* is a compact region which achieves the minimal $(n-1)$-dimensional Hausdorff measure of the boundary among all compact regions of the same n-dimensional Hausdorff measure. The former is a type of distinguished global geometric object characterized by one of the simplest local condition and the latter is a solution to one of the most fundamental geometric variation problems. Moreover, they are conceptually closely related in such a way that the *regular part* of the boundary of an isoperimetric region is a hypersurface of constant mean curvature.

In the simplest basic case of the Euclidean spaces, E^n, $n \geq 3$, the existence of *homothety transformations* enable us to *normalize* the constant of mean curvature to be equal to 1, i.e. $\operatorname{tr} \mathrm{II} = n-1$, and the spheres of unit radius are obvious examples of such soap bubbles in E^n. In fact, up to 1981, the round spheres were the only known examples of soap bubbles in E^n and the hyperbolic spaces H^n; and the beautiful uniqueness theorems of Hopf [Ho] and Alexandrov [Al-1], [Al-2] had, somehow, misled many geometers to believe that they are most likely the only possible ones (cf. the remark of Alexandrov in [Al-2]). The discovery of infinitely many congruent classes of *spherical* soap bubbles of mean curvature 1 in E^n for all $n \geq 4$ in 1982 [HTY-1], [HTY-2], [Hs-1] and the recent examples of Wente [We], Abresh [Ab], Kapouleas [Ka-1], [Ka-2] and Pinkall–Sterling [PS] on soap bubbles in E^3 with non-zero genus clearly indicate that there are abundant varieties of soap bubbles in E^n, $n \geq 3$, for us to discover and to understand. On the other hand, the situation of isoperimetric regions in E^n is quite different. Namely, it has already been neatly settled by the unique-existence theorem of Dinghas–Schmidt in the 1940's [DS].

Of course, the Euclidean spaces are just natural starting testing spaces for the study of both the soap bubbles and the isoperimetric regions. Therefore, it is rather natural to broaden the scope to include other fundamental Riemannian manifolds as the ambient spaces. For example, those simply connected, *non-positively curved* symmetric spaces are natural generalizations of E^n and H^n and they form an interesting family of testing spaces for the study of soap bubbles and isoperimetric regions. This family consists of those symmetric spaces of non-compact type and the products of