

ABSTRACT KAZHDAN-LUSZTIG THEORIES

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Abstract. In this paper, we prove two main results. The first establishes that Lusztig's conjecture for the characters of the irreducible representations of a semisimple algebraic group in positive characteristic is equivalent to a simple assertion that certain pairs of irreducible modules have non-split extensions. The pairs of irreducible modules in question are those with regular dominant weights which are mirror images of each other in adjacent alcoves (in the Jantzen region). Secondly, we establish that the validity of the Lusztig conjecture yields a complete calculation of all Yoneda Ext groups between irreducible modules having regular dominant weights in the Jantzen region. These results arise from a general theory involving so-called Kazhdan-Lusztig theories in an abstract highest weight category. Accordingly, our results are applicable to a number of other situations, including the Bernstein-Gelfand-Gelfand category for a complex Lie algebra and the category of modules for a quantum group at a root of unity.

A major unsolved problem in finite group theory centers on determining the characters and degrees of the irreducible modular representations of finite groups of Lie type in the defining characteristic. Lusztig took a significant step toward a solution in 1979 by formulating his celebrated conjecture [L1] for the characters of simple modules for semisimple algebraic groups. Since that time, mathematicians have devoted considerable effort to establishing this conjecture, which would completely solve the above problem as long as the characteristic is not too small relative to the root system. A similar conjecture, by Kazhdan and Lusztig [KL1], for the composition factor multiplicities of Verma modules for semisimple complex Lie algebras, has already been settled [BB], [BK]. For some time, we have worked to develop algebraic techniques in positive characteristic capturing some of the geometric methods used in the characteristic zero Lie algebra theory.

Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field k of positive characteristic p . For a dominant weight λ , let $L(\lambda)$ be the corresponding irreducible rational G -module of highest weight λ . This paper contains two main results. The first, given in Theorem 5.3, establishes that the Lusztig conjecture is equivalent to the simple assertion that $\text{Ext}_G^1(L(\lambda), L(\lambda')) \neq 0$ for p -regular

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