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FUNCTIONS WHICH OPERATE ON ALGEBRAS OF FOURIER MULTIPLIERS

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Abstract. We study functions which operate on a Banach space of bounded functions defined on a discrete space. As a consequence we characterize functions which operate on the algebra of the translation invariant operators from $L^p(G)$ to $L^2(G)$ for $1<p<2$ and for a compact abelian group G.

1. Introduction. Let G be a compact abelian group and $\hat{G}$ the dual group of G. Let $1\leq p, q\leq\infty$. A bounded operator T from $L^p(G)$ to $L^q(G)$ is called a (L^p,L^q)-multiplier if $TT_\gamma=T_\gamma T$ for every $\gamma\in G$, where $T_\gamma f(x)=f(x-\gamma)$. The set of all (L^p, L^q)-multipliers is denoted by $M(p, q)$. Since G is a compact abelian group the Fourier transform $\hat{T}$ for $T\in M(p, q)$ is a complex-valued bounded function defined on the discrete group $\hat{G}$. We denote $M(p, q)\hat{}=\{\hat{T}\colon T\in M(p, q)\}$. If $p\leq q$, then $M(p, q)$ is a Banach algebra and $M(p, q)\hat{}$ is a Banach algebra of bounded functions on $\hat{G}$. Let E be a space of complex-valued functions defined on a set X. We say a complex-valued function φ defined on a subset S of $\boldsymbol{C}$ operates on E if $\varphi\circ f\in E$ for every $f\in E$ such that $f(X)\subset S$.

The algebra $M(1, 1)$ is isometric and isomorphic to the algebra $M(G)$ of all the bounded regular Borel measures on G and the operating functions on $M(G)\hat{}$ is characterized by Kahane and Rudin [10]. The result is extended to the case of $p=q\neq 2$ by Igari [8]. Igari and Sato [9] consider the case of $1\leq p<q\leq\infty$. They prove, for example, that if $1\leq p<q\leq 2$ or $2\leq p<q\leq\infty$, n_0 is the smallest n integer such that $n\geq\beta_0=(1/q-1/2)/(1/p-1/q)$ or $n\geq\beta_0=(1/2-1/p)/(1/p-1/q)$ respectively, and φ_0 is a bounded function on $[-1, 1]$, then for any constants $\alpha_1, \alpha_2, \dots, \alpha_{n_0}$ the function

$$\varphi(t)=\alpha_1 t+\alpha_2 t^2+\cdots+\alpha_{n_0}t^{n_0}+|t|^{\beta_0+1}\varphi_0(t)$$

defined on $[-1, 1]$ operates on $M(p, q)\hat{}$. They also prove that if $1\leq p<2\leq q\leq\infty$, $\beta_1=\min\{(1/2-1/q)/(1/p-1/2), (1/p-1/2)/(1/2-1/q)\}$ and φ_0 is a bounded function on $[-1, 1]$, then for any constant α the function

$$\varphi(t)=\alpha t+|t|^{\beta_1+1}\varphi_0(t)$$

operates on $M(p, q)\hat{}$. The converse of the last result when G is the circle group is also proven by Igari and Sato [9]. One of the essential arguments they use in their proof

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