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# ON SHARPNESS, APPLICATIONS AND GENERALIZATIONS OF SOME CARLEMAN TYPE INEQUALITIES

LJUBOMIR T. DECHEVSKY AND LARS-ERIK PERSSON[1]

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**Abstract.** Carleman's inequality for Hilbert-Schmidt operators and its generalizations for Schatten-von Neumann operator ideals (see [7]) are shown to be sharp in a certain sense. Explicit classes of extremizing operators are found on which the generalized Carleman inequalities turn to asymptotic equalities. Applications are made to a priori estimation of the solutions of Fredholm and Volterra first- and second kind integral equations and to perturbation and error analysis. Some further generalizations are considered which extend the applications to singular integral equations, pseudo-differential equations and analytic functions of operator argument.

**1. Introduction.** In [7] upper bounds for the resolvent norm of a bounded linear operator $T$: $H \to H$, $H$ Hilbert space, were obtained in terms of the so-called generalized Carleman inequalities with minimal information about the spectrum.

In this note we prove that all results obtained in [7] are *sharp* in a certain sense. Simultaneously we point out a remarkable class of extremizing operators (which we call pre-orthogonal operators) on which the sharpness assertions are attained. Sharp constants are found, too (see Section 3).

We also include some *applications* of our results (see Section 4). In particular, some important applications are obtained for the classical problems of deriving a priori estimates for the solutions of the Fredholm and Volterra operator equations. As a further development of this idea we obtain a priori estimates for expressions of the type $(\lambda I - T_1)^{-1} f_1 - (\lambda I - T_2)^{-1} f_2$, which in the considered Hilbert-space case improve some corresponding earlier results in [24]. Of particular interest is the case where the compact operators are integral ones. In this case it is known that the Schatten-von Neumann quasinorms can be estimated very well by appropriate function quasinorms of the integral operator's kernel in Lebesgue and Besov spaces. These results, which, in their turn, may be regarded as generalizations of some classical results in the Hilbert-Schmidt theory, are in their final form due to Birman, Solomjak and Karadzhov (see [2], [3], [4], [12], [13], [14] and [15]). By combining these results with the main results in [7] it is possible to obtain useful a priori estimates for the solutions of classes of integral equations. In Section 4 we discuss briefly this possibility and present some examples of such applications (cf. also [8] and [9]). Another important application in this section

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