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# CELLS IN CERTAIN SETS OF MATRICES

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**Abstract.** We decompose the canonical bases for $q$-Schur algebras and the modified quantized enveloping algebras of type $A$ into two-sided cells in terms of some combinatorics on certain sets of matrices.

**Introduction.** Let $\mathscr{A}$ be an associative algebra over a field $K$ and $\mathscr{B}$ a basis for $\mathscr{A}$ as a $K$-vector space. Then $\mathscr{B}$ is divided into *cells*, by Lusztig [Lu2, 29.4], via the equivalence relations on $\mathscr{B}$ defined as follows:

If the elements $c_{b,b',b''} \in K$ with $b, b', b'' \in \mathscr{B}$ denote the structure constants of $\mathscr{A}$ i.e., they satisfy $bb' = \sum_{b'' \in \mathscr{B}} c_{b,b',b''} b''$, then we say for $b, b' \in \mathscr{B}$ that $b' \leq_L b$ (resp. $b' \leq_R b$) if there are sequences $b_1 = b, b_2, \ldots, b_n = b'$ and $\beta_1, \ldots, \beta_{n-1}$ in $\mathscr{B}$ such that $c_{\beta_i, b_i, b_{i+1}} \neq 0$ (resp. $c_{b_i, \beta_i, b_{i+1}} \neq 0$) for all $i = 1, \ldots, n-1$. These are preorders on $\mathscr{B}$. We define $\leq_{LR}$ to be the preorder on $\mathscr{B}$ generated by $\leq_L$ and $\leq_R$. For $x \in \{L, R, LR\}$ and $b, b' \in \mathscr{B}$ we say $b \sim_x b'$ if $b \leq_x b' \leq_x b$. Thus $\sim_L$, $\sim_R$ and $\sim_{LR}$ are equivalence relations on $\mathscr{B}$. The corresponding equivalence classes are called *left*, *right* and *two-sided cells* of $\mathscr{B}$ respectively.

In certain nice circumstances, cells are important in the study of representation theory and can be classified combinatorially. For example, if $\mathscr{A} = \mathscr{H}(W)$ is a Hecke algebra associated with a Coxeter group $W$ and $\mathscr{B}$ is the Kazhdan-Lusztig basis of $\mathscr{H}(W)$, then cells in the sense above are the Kazhdan-Lusztig cells (see [KL]). When $W$ is a Weyl group or an affine Weyl group of type $A$, K-L cells can be classified in terms of partitions, tableaux, Robinson-Schensted maps, etc. (see, e.g., [Sh]).

In this paper, we shall consider two more examples in the case of type $A$, namely, $\mathscr{A}$ is a $q$-Schur algebra $\mathscr{S}_q(n, r)$ or a modified quantized enveloping algebra $\dot{\boldsymbol{U}}$ of type $A$ and $\mathscr{B}$ is the K-L basis of $\mathscr{S}_q(n, r)$ (cf. [Du1]) or the canonical basis of $\dot{\boldsymbol{U}}$ (cf. [Lu2]). These bases are indexed by certain sets of matrices. So the cell decomposition of $\mathscr{B}$ induces a cell decomposition of these matrix sets. We shall give a combinatorial description for those two-sided cells.

In Section 1 we first generalize a result of Greene [G], which associates partitions to finite posets, to a result on a cerain matrix semigroup $M(n)$, namely one associates partitions to the matrices in $M(n)$. Thus, we decompose $M(n)$ into subsets in terms of partitions. Our main result is Theorem 2.1 which shows that these subsets are two-sided

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