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ON A CONSTRUCTION OF THE FUNDAMENTAL SOLUTION FOR THE FREE WEYL EQUATION BY HAMILTONIAN PATH-INTEGRAL METHOD —AN EXACTLY SOLVABLE CASE WITH "ODD VARIABLE COEFFICIENTS"

Dedicated to Professor Takeshi Kotake on his retirement from Tôhoku University

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Abstract. A fundamental solution for the free Weyl equation is easily constructed using the Clifford relation of the Pauli matrices. But, we insist on Feynman's idea of representing a fundamental solution using classical objects. To do this, we first reformulate the usual matrix-valued Weyl equation on the ordinary Euclidian space to the "non-commutative scalar"-valued equation on the superspace, called the super Weyl equation. Then, we may find the classical mechanics corresponding to that super Weyl equation. Using analysis on the superspace, we may associate the classical Hamiltonian with that super Weyl equation. From this mechanics, we define phase and amplitude functions which are solutions of the Hamilton-Jacobi and continuity equations, respectively. Moreover, they are exactly solvable. Then, we define a Fourier integral operator with phase and amplitude given by those functions, which gives a solution to the initial value problem of that super Weyl equation. The method and idea developped here, may be applied not only to the Pauli, Weyl or Dirac equations but also to any system of P.D.E's.

1. Introduction and the result.

Let $\psi(t, q): \boldsymbol{R} \times \boldsymbol{R}^3 \to \boldsymbol{C}^2$ satisfy

$$(1.1)\qquad \begin{cases} i\hbar \dfrac{\partial}{\partial t}\psi(t, q) = \boldsymbol{H}\psi(t, q), & \boldsymbol{H} = -ic\hbar\boldsymbol{\sigma}_j \dfrac{\partial}{\partial q_j}, \\ \psi(0, q) = \underline{\psi}(q). \end{cases}$$

Here, $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q))$, c and $\hbar$ are positive constants, the summation with respect to $j = 1, 2, 3$ is abbreviated. And the Pauli matrices $\{\boldsymbol{\sigma}_j\}$ are 2×2 matrices satisfying the following relations ($\boldsymbol{I}_m$ stands for the $m \times m$ identity matrix):

$$(1.2)\qquad \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k + \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j = 2\delta_{jk}\boldsymbol{I}_2 \quad \text{for} \quad j, k = 1, 2, 3, \quad \text{(Clifford relation)}$$

$$(1.3)\qquad \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_3, \quad \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_1, \quad \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = i\boldsymbol{\sigma}_2,$$

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