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ON SYMMETRIES OF CONSTANT MEAN CURVATURE SURFACES, PART I: GENERAL THEORY

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Abstract. We start the investigation of immersions of a simply connected domain into three dimensional Euclidean space which have constant mean curvature (CMC-immersions), and allow for a group of automorphisms of the domain which leave the image invariant. This leads to a detailed description of symmetric CMC-surfaces and the associated symmetry groups.

1. Introduction. This is the first of two parts of a note in which we start the investigation of conformal CMC-immersions $\Psi:\mathscr{D}\to\boldsymbol{R}^3$, $\mathscr{D}$ an open, simply connected subset of $\boldsymbol{C}$, which allow for groups of spatial symmetries

$$\mathrm{Aut}\,\Psi(\mathscr{D})=\{\tilde{T}\text{ proper Euclidean motion of }\boldsymbol{R}^3\mid\tilde{T}\Psi(\mathscr{D})=\Psi(\mathscr{D})\}\,.$$

More precisely (see the definition in Section 2), we consider a Riemann surface M with universal covering $\pi:\mathscr{D}\to M$, and a conformal CMC-immersion $\Phi: M\to\boldsymbol{R}^3$ with nonzero mean curvature, such that $\Phi\circ\pi=\Psi$. Then we consider the groups

$$\mathrm{Aut}\,\mathscr{D}=\{g:\mathscr{D}\to\mathscr{D}\text{ biholomorphic}\}\,,$$

$$\mathrm{Aut}\,M=\{g:M\to M\text{ biholomorphic}\}\,,$$

$$\mathrm{Aut}_\pi\,\mathscr{D}=\{g\in\mathrm{Aut}\,\mathscr{D}\mid\text{there exists }\hat{g}\in\mathrm{Aut}\,M:\pi\circ g=\hat{g}\circ\pi\}\,,$$

$$\mathrm{Aut}_\Phi\,M=\{\hat{g}\in\mathrm{Aut}\,M\mid\text{there exists }\tilde{T}\in\mathrm{Aut}\,\Psi(\mathscr{D}):\Phi\circ\hat{g}=\tilde{T}\circ\Phi\}\,,$$

and

$$\mathrm{Aut}_\Psi\,\mathscr{D}=\{g\in\mathrm{Aut}\,\mathscr{D}\mid\text{there exists }\tilde{T}\in\mathrm{Aut}\,\Psi(\mathscr{D}):\Psi\circ g=\tilde{T}\circ\Psi\}\,.$$

There are many well-known examples of CMC-surfaces with large spatial symmetry groups. The classic Delaunay surfaces (see [2]) have a nondiscrete group $\mathrm{Aut}\,\Psi(\mathscr{D})$ containing the group of all rotations around their generating axis. Other examples are the Smyth surface [9], which were visualized by D. Lerner, I. Sterling, C. Gunn and U. Pinkall. These surfaces have an $(m+2)$-fold rotational symmetry in $\boldsymbol{R}^3$, where the axis of rotation passes through the single umbilic of order m. More recent is the large class of examples provided by Große-Brauckmann and Polthier (see e.g. [4], [5]) of

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