THE FABER-KRAHN TYPE ISOPERIMETRIC INEQUALITIES FOR A GRAPH

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Abstract. In this paper, a graph theoretic analog to the celebrated Faber-Krahn inequality for the first eigenvalue of the Dirichlet problem of the Laplacian for a bounded domain in the Euclidean space is shown. Namely, the optimal estimate of the first eigenvalue of the Dirichlet boundary problem of the combinatorial Laplacian for a graph with boundary is given.

1. Introduction. The celebrated Faber-Krahn inequality is stated as follows (see [1], [2]):

FABER-KRAHN THEOREM. *Let $\lambda_1(\Omega)$ be the first eigenvalue of the Dirichlet Laplacian for a bounded domain Ω in $\mathbf{R}^n$. If $\mathrm{Vol}(\Omega) = \mathrm{Vol}(\Omega^*)$, where Ω^* is a ball in $\mathbf{R}^n$, then*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*),$$

and the equality holds if and only if Ω is congruent to Ω^*.

In this paper, we show an analog of the Faber-Krahn theorem for a graph. A graph is a collection of vertices together with a collection of edges joining pairs of vertices. Let us take a connected graph with boundary, $G = (V \cup \partial V, E \cup \partial E)$ (see the definition in Section 2). We consider the Dirichlet boundary problem of the combinatorial Laplacian Δ on G:

$$\begin{cases} \Delta f(x) = \lambda f(x), & x \in V, \\ f(x) = 0, & x \in \partial V. \end{cases}$$

Let us denote the eigenvalues for this problem by

$$0 < \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_k(G),$$

where k is the number of vertices in V. We call $\lambda_1(G)$ the *first eigenvalue* of G.

We give the following two examples (1), (2) of graphs with boundary: Here we denote by white (resp. black) circles, vertices in V (resp. ∂V) and by solid (resp. dotted) lines, edges in E (resp. ∂E).