Tohoku Math. J.
51 (1999), 237–265

UNITARY TORIC MANIFOLDS, MULTI-FANS AND EQUIVARIANT INDEX

Dedicated to Professor Akio Hattori on his seventieth birthday

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(Received February 17, 1998, revised July 31, 1998)

Abstract. We develop the theory of toric varieties from a topological point of view using equivariant cohomology. Indeed, we introduce a geometrical object called a unitary toric manifold and associate a combinatorial object called a multi-fan to it. This generalizes (in one direction) the well-known correspondence between a compact nonsingular toric variety and a (regular) fan. The multi-fan is a collection of cones which may overlap unlike a usual fan. It turns out that the degree of the overlap of cones is essentially the Todd genus of the unitary toric manifold. Since the Todd genus of a compact nonsingular toric variety is one, this explains why cones do not overlap in a usual fan. A moment map relates a unitary toric manifold equipped with an equivariant complex line bundle to a "twisted polytope", and the equivariant Riemann-Roch index for the equivariant line bundle can be described in terms of the moment map. We apply this result to establish a generalization of Pick's formula.

Introduction. The theory of toric varieties says that there is a one-to-one correspondence between toric varieties (an object in algebraic geometry) and fans (an object in combinatorics). This correspondence often brought new insights to combinatorics from algebraic geometry, and vice versa (see [2], [4], [15]).

A compact nonsingular toric variety is called a toric manifold and the corresponding fan is called regular. Toric manifolds are well studied among toric varieties and play an important role in the theory of toric varieties. In this paper we develop the correspondence between toric manifolds and regular fans from a topological point of view. In fact, our geometrical object called a *unitary toric manifold* constitutes a much wider class than that of toric manifolds. A unitary (resp. almost complex) toric manifold M is a compact unitary (resp. almost complex) manifold with an action of a compact torus T having nonempty isolated fixed points, where $2\dim_{\boldsymbol{R}} T=\dim_{\boldsymbol{R}} M$. The Todd genus of a unitary (resp. almost complex) toric manifold takes any (resp. positive) integer, while that of a toric manifold is one.

To a unitary toric manifold M we associate a combinatorial object Δ_M called the *multi-fan* of M using equivariant cohomology. To this end, closed connected real codimension two submanifolds M_i $(i=1,\dots,d)$ of M, left fixed by certain circle subgroups, play an essential role. Each M_i defines an element ξ_i in the equivariant cohomology $H^2_T(M;\boldsymbol{Z})$ through Poincaré duality and ξ_i's are used to associate an element $v_i \in H_2(BT;\boldsymbol{Z})$ to each M_i. To each subset $I\subset\{1,\dots,d\}$ such that $\bigcap_{i\in I} M_i \neq \emptyset$,

1991 *Mathematics Subject Classification.* Primary 57S15; Secondary 14M25, 57S25.