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THE MINIMAL MODEL THEOREM FOR DIVISORS OF TORIC VARIETIES

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Abstract. The minimal model conjecture says that if a proper variety has non-negative Kodaira dimension, then it has a minimal model with abundance and if the Kodaira dimension is $-\infty$, then it is birationally equivalent to a variety which has a fibration with the relatively ample anti-canonical divisor. In this paper, first we prove this conjecture for a Δ-regular divisor on a proper toric variety by means of successive contractions of extremal rays and flips of the ambient toric variety. Then we prove the main result: for such a divisor with the non-negative Kodaira dimension there is an algorithm to construct concretely a projective minimal model with abundance by means of "puffing up" the polytope.

Introduction. Let k be an algebraically closed field of arbitrary characteristic. Varieties in this paper are all defined over k. Let X be a proper algebraic variety. A proper algebraic variety Y is called a minimal model of X, if (1) Y is birationally equivalent to X, (2) Y has at worst terminal singularities and (3) the canonical divisor K_Y is nef. A minimal model Y is said to have abundance if the linear system $|mK_Y|$ is base point free for sufficiently large m. The minimal model conjecture states: an arbitrary proper variety with $\kappa \geq 0$ has a minimal model with abundance while an arbitrary proper variety with $\kappa = -\infty$ has a birationally equivalent model Y with at worst terminal singularities and a fibration $Y \to Z$ to a lower dimensional variety with $-K_Y$ relatively ample.

The conjecture is classically known to hold in the 2-dimensional case. In the 3-dimensional case the conjecture for $k = \boldsymbol{C}$ is proved by Mori [4] and Kawamata [3], while it is not yet proved in higher dimension. As a special case of higher dimension, Batyrev [1] proved, among other results, the existence of a minimal model for a Δ-regular anti-canonical divisor of a Gorenstein Fano toric variety $T_N(\Delta)$.

In this paper, first in Section 1 we prove the minimal model conjecture for every Δ-regular divisor X on a toric variety of arbitrary dimension by means of successive contractions of extremal rays and flips which are introduced by Reid [7]. By Bertini's theorem, for a field k of characteristic 0, the minimal model conjecture thus holds for a general member of a base point free linear system on a proper toric variety over k. An important point of this part is providing with a technical statement Corollary 1.17 which is used in the following sections. Then in Sections 2 and 3 we prove the main result: for a Δ-regular divisor with $\kappa \geq 0$, there exists an algorithm to construct concretely a projective minimal model with abundance by means of "puffing up" the polytope corresponding to the adjoint divisor. The advantage of

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