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GLOBALLY DEFINED LINEAR CONNECTIONS ON THE REAL LINE AND THE CIRCLE

Dedicated to Professor Thomas James Willmore on his seventy-seventh birthday

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Abstract. We classify globally defined linear connections on the real line as well as on the circle up to diffeomorphisms. We also prove that such connections can be realized by affine immersions into the affine plane.

1. Linear connections on $\boldsymbol{R}^1$ and S^1. On the real line $\boldsymbol{R}^1$ with its usual coordinate system $\{x\}$, we denote by d/dx the vector field (or its value at a point). A linear connection ∇ can be determined by a function Γ such that

$$\nabla_x \frac{d}{dx} = \Gamma(x) \frac{d}{dx}, \tag{1}$$

where ∇_x denotes covariant differentiation relative to d/dx.

Since $\boldsymbol{R}^1$ is 1-dimensional, every connection ∇ on $\boldsymbol{R}^1$ must be locally flat. This means that for each point x_0, there is a neighborhood with a flat local coordinate system, $\{\bar{x}\}$, relative to which we have

$$\nabla_{\bar{x}} \frac{d}{d\bar{x}} = 0 . \tag{2}$$

We deal with the question: How do we find such a local coordinate system $\{\bar{x}\}$?

This is related to the question of finding a geodesic relative to ∇, that is, of solving the equation

$$\frac{d^2 x}{dt^2} + \Gamma(x(t)) \left(\frac{dx}{dt} \right)^2 = 0 , \tag{3}$$

with the initial conditions, for convenience, say,

$$x(0) = 0 , \quad \left(\frac{dx}{dt} \right)(0) = 1 . \tag{4}$$

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